Central Limit theorem

Suppose $X_1, X_2, \ldots$ are identical, independent random variables w/ mean $\mu = E[X_i]$ variance $\sigma^2 = Var(X_i)$

Then

$$\lim_{n\to\infty} P\left(\frac{\left(\sum_{i=1}^{n} X_i\right) - n\mu}{\sigma\sqrt{n}} \leq a\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}\, dx = P(Z \leq a)$$

Z is "unit" normal random variable

p.d.f. $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$E[X+Y] = E[X] + E[Y]$ $\quad\quad \mu_{X+Y} = \mu_X + \mu_Y$

$E[X - \mu_X] = E[X] - E[\mu_X] = \mu_X - \mu_X = 0$

$Var(X+Y) = Var(X) + Var(Y)$

(if X & Y independent)

$$\mathrm{Var}(\lambda X) = \mathrm{Cov}(\lambda X, \lambda X) = \lambda^2 \mathrm{Cov}(X,X)$$
$$= \lambda^2 \mathrm{Var}(X).$$

So $\dfrac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}}$ is a random var w/
mean 0
variance 1

Basic tool: Moment generating function

Recall: $M_X(t) = E[e^{tX}]$

$t=2$ $E[e^{2X}]$ = some # $\quad M_X(2)$

$t=3$ $E[e^{3X}]$ . . . $\quad M_X(3)$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

cont. case

Compare $\hat{f}(t) = \int_{-\infty}^{\infty} e^{-2\pi i x t} f(x)\,dx$

i.e. $M_X(-2\pi i t) = \hat{f}(t)$

Important property: $f(x) = \int_{-\infty}^{\infty} e^{2\pi i x t} \hat{f}(t)\,dt$

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i x t} M_X\left(\frac{t}{-2\pi i}\right) dt$$

$$\left(e^{i\theta} = \cos\theta + i\sin\theta \right)$$

Properties of $M_X(t)$

- Determines X i.e. $M_X(t) = M_Y(t) \Leftrightarrow X = Y$
- X & Y indep $M_{X+Y}(t) = M_X(t)\,M_Y(t)$
- $M'(0) = E[X]$ $M''(0) = E[X^2], \ldots$

  $M(0) = E[e^0] = 1$

Power series expansion:

$$M(t) = M(0) + M'(0)t + \frac{1}{2}M''(0)t^2 + \ldots$$

$$= \sum \frac{M^{(n)}(0)}{n!} t^n = \sum_{i=0}^{\infty} \frac{E[X^n]}{n!} t^n$$

Step 1 of CLT proof

$M_Z(t)$ $\qquad$ Z p.d.f. $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$M_Z(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2 + tx}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2-2xt)/2}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-t)^2+t^2}{2}}\, dx$$

$$= e^{t^2/2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2}\, dx}$$

p.d.f for normal var w/
mean t
var 1

$$M_Z(t) = e^{t^2/2}$$

Suppose $X_1, X_2, \ldots$ are random vars identical, independent w/ mean 0 & Var 1

$$Y_n = \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}}$$

let $M_n(t)$ = MGF of Y_n.

if $M(t) = M_{X_i}(t)$

$$M_n(t) = M(t/\sqrt{n})^n$$

$$\boxed{\begin{aligned} M_{\lambda X}(t) &= E[e^{t\lambda X}] \\ &= M_X(\lambda t) \end{aligned}}$$

$\lim_{n \to \infty} M_n(t)$ (will show $= e^{t^2/2}$) then will be done.

$$\lim_{n \to \infty} M(t/\sqrt{n})^n$$

$$L(t) = \log M(t)$$

$$L(0) = \log M(0) = \log 1 = 0$$

$$L'(t) = \frac{M'(t)}{M(t)} \qquad L'(0) = \frac{M'(0)}{M(0)} = \frac{0^{\leftarrow \mu}}{1} = 0$$

$$L''(t) = \frac{M(t)\,M''(t) - M'(t)^2}{M(t)^2}$$

$$L''(0) = \frac{M(0)\,1 - 0}{1^2} = 1$$

$$\log\left(\lim_{n\to\infty} M(t/\sqrt{n})^n\right) = \lim_{n\to\infty} n \log M(t/\sqrt{n})$$

$$= \lim_{n\to\infty} \frac{L(t n^{-1/2})}{n^{-1}} \qquad \text{"}\tfrac{0}{0}\text{"}$$

L'Hopital

$$= \lim_{n\to\infty} \frac{L'(t n^{-1/2})\, t\, (-\tfrac{1}{2}) n^{-3/2}}{-n^{-2}}$$

$$= \lim_{n\to\infty} \frac{L'(t n^{-1/2})\, t}{2 n^{-1/2}} \qquad \text{"}\tfrac{0}{0}\text{"}$$

L'Hop

$$= \lim_{n\to\infty} \frac{L''(t n^{-1/2})\, t\, \cancel{(-\tfrac{1}{2})}\, \cancel{n^{-3/2}}\, t}{2 \cancel{(-\tfrac{1}{2})}\, \cancel{n^{-3/2}}}$$

$$= \lim_{n\to\infty} \frac{L''(t n^{-1/2})\, t^2}{2} = L''(0)\,\frac{t^2}{2} = \frac{t^2}{2}$$

$$\log \lim_{n\to\infty} M(t/\sqrt{n})^n = t^2/2$$

$$\underbrace{\lim_{n\to\infty} M_n(t)}_{} = \underbrace{e^{t^2/2}}_{}$$

$$\lim_{n\to\infty} \frac{\sum X_i}{\sqrt{n}} = Z$$

"Convergence" $\Rightarrow \lim_{n\to\infty} P\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}} \le a\right) = P(Z \le a)$

Consequently, if $X_1, X_2, \ldots$ are general identical, indep...

mean μ var σ^2

$$P\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right)$$

$$=$$

$$P\left(\frac{\left(\frac{(X_1-\mu)}{\sigma} + \frac{(X_2-\mu)}{\sigma} + \cdots + \frac{(X_n-\mu)}{\sigma}\right)}{\sqrt{n}} \le a\right)$$

$$=$$

$$P(Z \le a) \quad \square.$$