- Probability density & distribution functions
- Expected Value & Variance
- Examples: Bernoulli/binomial, Poisson.

<u>Recall</u>: random variable is a function

$$X : S \longrightarrow \mathbb{R} \qquad S = \text{sample space.}$$

useful to think of X "independently of S"

$P(X=5) \qquad P(X \leq 7)$ (last time)

In the case where X only obtains ~~finitely~~ countable many values

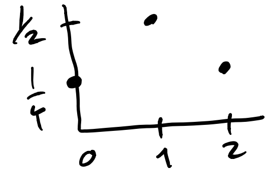
represent probabilities visually:

define: probability density function (PDF)

$$\boxed{p(a) = P(X=a)}$$

2 coins, $X = \#$ heads when flipped

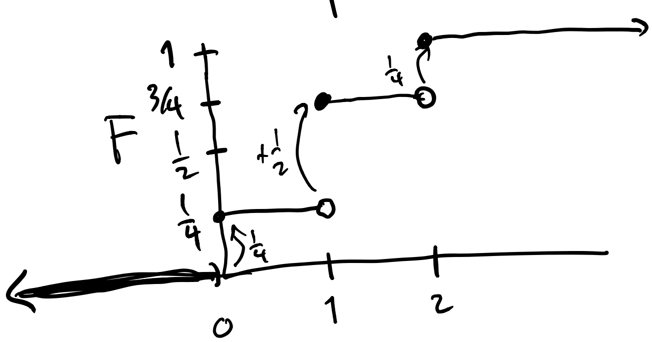
$P(X=0) = \frac{1}{4} = P(X=2) \qquad P(X=1) = \frac{1}{2}$



also can represent information by cumulative distribution function

Define the (cumulative) distribution function (CDF) as $F(a) \equiv P(X \leq a)$

above example $X = \#$ heads in 2 flips



Rem: always increasing

Expected Value (assume only countably many possible values for X)

Def $$E[X] = \sum_{\substack{x \text{ s.t.} \\ p(x) > 0}} x\,p(x)$$

i.e. if $a_1, a_2, \ldots$ are all possible values X can take

$$E[X] = \sum_i a_i p(a_i) = \sum_i a_i P(X = a_i)$$

Think of this as "average value" of X after large # of measurements/experiments.

$E[X]$ $\quad X = \#$ heads in 2 flips

$$= 0\,P(X=0) + 1\,P(X=1) + 2\,P(X=2)$$

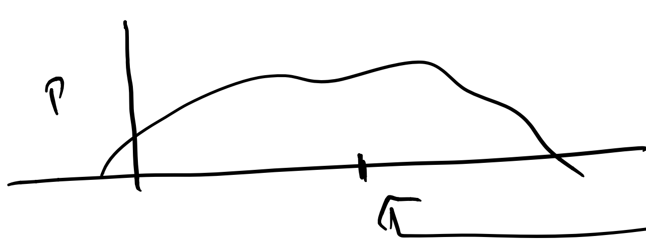
$$1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{4} = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

game: w/ prob $1/1000$ win \$1,000,000

prob $999/1000$ lose \$1

X = winnings $\quad E[X] = (-1)\,\tfrac{999}{1000} + (1{,}000{,}000)\,\tfrac{1}{1000}$

$$\approx 1000 - 1 \approx 1{,}000$$

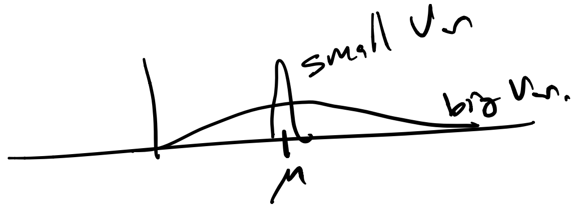


$E[X]$ "center of gravity"

Variance:

$$\mu = E[X]$$

$$E[(X-\mu)^2] \equiv Var(X) \longleftrightarrow \text{moment of inertia.}$$



Example Binomial random variables

Bernoulli Random Variable

$$\left.\begin{aligned} P(X=0) &= 1-p \\ P(X=1) &= p \end{aligned}\right] \text{ X is called a Bernoulli random variable}$$

Similarly, can repeat n times and let

X = sum of n results.

$\Longrightarrow$ Bernoulli Random variable w/ parameters (n, p).

$$\boxed{P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}}$$

(possible vals are $0, \ldots, n$)

Poisson variable

X binomial random var w/ params (n, p)

$n \gg 0$ want np to be "regular size"

" λ

$n \gg \lambda, i$

"i regular size"

$P(X = i)$

$p = \frac{\lambda}{n}$

$$= \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \frac{n!}{i!(n-i)!} \frac{\lambda^i}{n^i} \left(1 - \lambda/n\right)^{n-i}$$

$$= \frac{n!}{i!(n-i)!} \frac{\lambda^i}{n^i} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}$$

$\xrightarrow{n\to\infty} 1$ $\xrightarrow{n\to\infty} 1$ $\xrightarrow{n\to\infty} e^{-\lambda}$

$$P(X=i) \approx \frac{\lambda^i}{i!} e^{-\lambda}$$

<u>Def</u>* Poisson Variable to be one w/

$$P(X=i) = \lambda^i/i! \; e^{-\lambda}$$