Math 477, Lecture 5 class work

Name: ____

Net ID: ____

1. Suppose that 4 student homeworks are collected and passed back at random to the students. What is the probability that no one gets the correct assignment back?

   Let H_i represent the event that the i'th person receives the correct homework. The event that no one receives their homework is $(H_1 \cup H_2 \cup H_3 \cup H_4)^c$, and so we can calculate $P(H_1 \cup H_2 \cup H_3 \cup H_4)^c = 1 - P(H_1 \cup H_2 \cup H_3 \cup H_4)$.

   By the inclusion/exclusion principle, we have

$$\begin{aligned} P(H_1 \cup H_2 \cup H_3 \cup H_4) = & P(H_1) + P(H_2) + P(H_3) + P(H_4) \\ & - P(H_1H_2) - P(H_1H_3) - P(H_1H_4) - P(H_2H_3) - P(H_2H_4) - P(H_3H_4) \\ & + P(H_1H_2H_3) + P(H_1H_2H_4) + P(H_1H_3H_4) + P(H_2H_3H_4) \\ & - P(H_1H_2H_3H_4) \end{aligned}$$

   There are 3! ways that H_i can happen (these are the ways to distribute everyone else's homework, after we know that i gets their own homework), 2! ways that H_iH_j can happen (if $i \neq j$), etcetera. Therefore we get

$$P(H_1 \cup H_2 \cup H_3 \cup H_4) = 4\frac{3!}{4!} - 6\frac{2!}{4!} + 4\frac{1!}{4!} - \frac{0!}{4!} = 1 - 1/2 + 1/6 - 1/24 = (24 - 12 + 4 - 1)/24 = 15/24$$

   and so the answer is $1 - 15/24 = 7/24$.

2. Suppose that 30% of the population is accident prone and will has a 40% chance of having an accident in any given year, while 70% of the population has only a 5% chance of having an accident in a given year. Suppose that a certain person has an accident in one year.

   (a) What's the probability that the person is accident prone?

   Let A be the event of having an accident in the first year, and T the event that the person is accident prone. We want to calculate $P(T|A)$. We have:

$$\begin{aligned} P(T|A) = P(AT)/P(A) = \frac{P(A|T)P(T)}{P(A)} = \frac{P(A|T)P(T)}{P(A|T)P(T) + P(A|T^c)P(T^c)} = \\ \frac{(4/10)(3/10)}{(4/10)(3/10) + (1/20)(7/10)} = \frac{12/100}{12/100 + 7/200} = \frac{24/200}{31/200} = 24/31 \end{aligned}$$

(b) What's the probability that the same person will have an accident the following year as well?

Let B be the event that the person has an accident in the second year. We have $P(B|A) = P(AB)/P(A)$, so we calculate $P(AB)$ and $P(A)$

$$P(A) = P(A|T)P(T)+P(A|T^c)P(T^c) = (4/10)(3/10)+(1/20)(7/10) = 12/100+7/200 = 31/200$$

Now, since an accident prone driver has a 30% chance of having an accident each year, and a non-accident prone drive has a 5% chance of having an accident each year, we have:

$$\begin{aligned} P(AB) = P(AB|T)P(T) + P(AB|T^c)P(T^c) &= (4/10)^2(3/10) + (1/20)^2(7/10) \\ &= 48/1000 + 7/4000 = (192+7)/4000 = 199/4000 \end{aligned}$$

so $P(B|A) = P(AB)/P(A) = (199/4000)/(31/200) = (200)(199)/(31)(4000) = 199/(31 \times 20) = 199/690 \sim .29$.

3. Suppose we have a blood test for a given disease. If you are sick, there is a 95% chance that the blood test will detect it. If you are well, there is a 1% chance that the blood test will falsely show that you are sick. Suppose only 0.1% of the population has the disease. If a random person is given the test, and it results in showing that they are sick, what is the probability that they are actually sick?

4. Suppose we have a container full of batteries, where type A batteries have a 10% chance of failing after 10 hours of use, type B batteries have a 20% chance of failing after 10 hours of use, and type C batteries have a 30% chance of failing after 10 hours of use. Suppose that there are equal numbers of each type of battery and one is chosen at random.

(a) What is the probability that the battery will fail after 10 hours?

Let A, B and C represent the events that a chosen battery is type A, B or C respectively. Let F be the event that the battery fails after 10 hours. If S is the whole sample space, then $A \cup B \cup C = S$ since every battery is either type A, B or C.
We have

$$P(F) = P(FS) = P(F(A \cup B \cup C)) = P(AF \cup BF \cup CF) = P(AF) + P(BF) + P(CF)$$

by one of the axioms of probability theory, since a battery can only be of one type, and hence the sets AF, BF, CF can't intersect. But, we also have

$$P(AF) = P(F|A)P(A) = (1/10)(1/3),\ P(BF) = P(F|B)P(B) = (2/10)(1/3),\ P(CF) = (3/10)(1/3)$$

and so

$$P(F) = P(AF) + P(BF) + P(CF) = 1/30 + 2/30 + 3/30 = 6/30 = 1/5.$$

(b) If the battery does not fail after 10 hours of use, what is the probability that is was a type A battery?

Using the above notation, we want to calculate $P(A|F)$. We have

$$P(A|F^c) = P(AF^c)/P(F^c) = \frac{P(F^c|A)P(A)}{1-P(F)} = \frac{(9/10)(1/3)}{4/5} = (9/30)(5/4) = 3/8$$