Plot:
- Geometry of solutions to polynomial equations can be encoded in (systems of) commutative rings
- Comm. rings can be interpreted as rings of "regular" functions on some special top. spaces.

Mantra: All comm. rings are rings of functions.

Ring of Functions encode geometry.

Rings = Geometry.

Ex: M C^∞ mfld, rule $U \longmapsto C^\infty(U)$ rings

encodes smooth mfld structure.

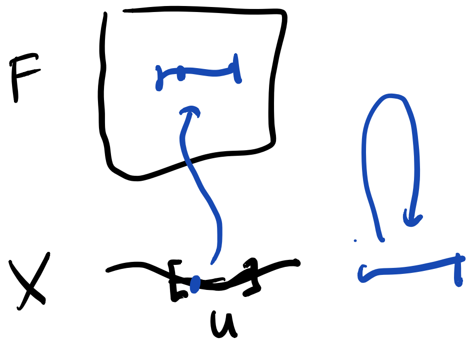
"**Ex:**" $X = \{*\}$ some ring R

Sheaves

Example X top space, $F \xrightarrow{\pi} X$ cont map

(F is an X-space) define

$S_\pi(U) = \{ s : U \to F \mid \pi s = id_U \}$



If $V \subset U$, have a natural map

$$S_\pi(U) \longrightarrow S_\pi(V)$$

$$\mathrm{res}_{S_\pi} : s \longmapsto s|_V$$

$$|_V$$

s.t. • $W \subset V \subset U$ then

$$(s|_V)|_W = s|_W$$

• $s|_U = s \quad (s \in S_\pi(U))$

• If U_i a cover of U then we have an equalizer diagram

$$S_\pi(U) \to \prod_i S_\pi(U_i) \rightrightarrows \prod_{i,j} S_\pi(U_i \cap U_j)$$

$$(s_i) \rightrightarrows (t_{ij})$$

$t_{ij} = s_j|_{U_i \cap U_j}$ $\qquad$ $t_{ij} = s_i|_{U_i \cap U_j}$

(Recall: we say $A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$ an eq. diagram if f maps A bijectively to $\{b \in B \mid gb = hb\}$)

Def A sheaf is a functor $\mathcal{F} : Op(X)^{op} \to Set$ (or some other cat $\mathcal{C}$)
such that conditions above hold

Def A presheaf is a functor $\mathcal{F} : Op(X)^{op} \to Set$ or $\mathcal{C}$
s.t. $\mathcal{F}(\emptyset) = \{*\}$
(terminal)

Def A separated presheaf is a functor s.t.
$\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective for all covers $\{U_i\}$ of U.

Def A geometric sheaf is a sheaf of the form S_π
some X-space $F \xrightarrow{\pi} X$.

Thm ("Cayley theorem of sheaves") All sheaves of sets are isomorphic to geometric sheaves.

We have "inclusions": $Shv(\mathcal{C}) \hookrightarrow SepPreShv(\mathcal{C}) \hookrightarrow PreShv(\mathcal{C}) \hookrightarrow Functor(\mathcal{C})$

$Functors \equiv Fun(Op(X)^{op}, \mathcal{C})$
category, morphisms = natural transformations

Def: morphism of Shv, SepPreShv, ... = nat trans. as above.

Cases of interest: $\mathcal{C}$ = Sets, Abgp, CommRngs

In case I forget: ring = comm. rng w/ unit, associative.

If $f: X \to Y$ cont map of top spaces

$f^p: \mathrm{Fun}(Y) \longrightarrow \mathrm{Fun}(X)$ $\qquad$ $\mathrm{Fun}(X) = \mathrm{Fun}(\mathrm{Op}(X)^{op}, \mathcal{C})$

$f_p: \mathrm{Fun}(X) \longrightarrow \mathrm{Fun}(Y)$

$$f_p(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)) \quad \text{natural restriction}$$

$$f^p(\mathcal{F})(U) = \varinjlim_{V \supset f(U)} \mathcal{F}(V) \quad \text{natural restriction.}$$

take presheaves to presheaves.

Note: A presheaf on a space w/ a single point is equiv. to an object in $\mathcal{C}$ $\qquad X = \{x\}$

$\mathrm{PreShv}(X) \simeq \mathcal{C}$ equiv. of cats.

Def if $\mathcal{F}$ a presheaf (or functor) and $x \in X$

$\mathcal{F}_x$ = (the presheaf assoc. to) $(i_x)^p(\mathcal{F})$

"stalk of $\mathcal{F}$ at x" $\qquad i_x: \{x\} \hookrightarrow X$

i.e. $\mathcal{F}_x = \varinjlim_{V \ni x} \mathcal{F}(V)$ given $t \in \mathcal{F}(V)$ $x \in V$
image of t in $\mathcal{F}_x$ is denoted t_x

Def Étale space of a presheaf

$\mathcal{F}$ a presheaf define $\acute{E}t(\mathcal{F}) = \{ (s,x) \mid x \in X, s \in \mathcal{F}_x \}$

$$\begin{array}{ccc} (s,x) & & \acute{E}t(\mathcal{F}) \\ \downarrow & & \downarrow \\ x & & X \end{array}$$

basis for top of $\acute{E}t(\mathcal{F})$ consisting of

$U_t = \{ (s,x) \mid x \in U, \ s = \text{im of } t \text{ in } \mathcal{F}_x \}$

$U \subset X$ open $t \in \mathcal{F}(U)$

Can check: get a natural map of presheaves

$$\mathcal{F} \longrightarrow S_{\acute{E}t(\mathcal{F})}$$

$$\begin{array}{ccl} \mathcal{F}(U) & \longrightarrow & S_{\acute{E}t(\mathcal{F})}(U) \\ t & \longmapsto & \big(U \to \acute{E}t(\mathcal{F}) \big) \\ & & \quad x \longmapsto (x, t_x) \end{array}$$

Thm: if $\mathcal{F}$ is a sheaf this is bijective!
(iso of presheaves!)

$$\mathrm{Hom}_{pre}(\mathcal{F}, \underset{\text{sheaf}}{\mathcal{G}}) = \mathrm{Hom}_{shv}(S_{\acute{E}t(\mathcal{F})}, \mathcal{G})$$

"univ. prop."

Def $\mathcal{F}^+ = S_{\acute{E}t(\mathcal{F})}$ "sheafification"

sheafification is left adjoint to forgetfull

Cor: if $\mathcal{F}$ is a sheaf, then $\mathcal{F} \simeq S_{\acute{E}t(\mathcal{F})}$

so is iso. to a geometric

We care (mostly) about sheaves, not presheaves.

If $f: X \to Y$ cont. map of top spaces,

then we get $f_*: Shv(X) \to Shv(Y)$

$f^{-1}: Shv(Y) \to Shv(X)$

$f_* = f_p \qquad f^{-1} = (\ )^+ \circ f^p$

note: if $f: \underset{\text{open}}{U \hookrightarrow X}$ inclusion then $f^{-1} = f^p$

in general if $f: Z \hookrightarrow X$ inclusion, notation

$f^{-1}\mathcal{F} = \mathcal{F}|_Z$

Def $f : \mathcal{F} \to \mathcal{G}$ of presheaves, we say f is inj, surj if

$\forall U$, $\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)$ are inj or surj.

Def $f : \mathcal{F} \to \mathcal{G}$ of sheaves, we say f is inj/surj if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ are inj/surj all $x \in X$

$\mathcal{C} = \mathbf{Abgp}$ we talk about $\mathbf{AbPre}$ $\mathbf{AbShv}$

Def presheaf $\mathrm{im}(f)$: $\mathrm{preim}(f)(U) = \mathrm{im}(f(U))$

$$\mathcal{F} \xrightarrow{f} \mathcal{G}$$

$$\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)$$

presh $\ker(f)$: $\mathrm{preker}(f)(U) = \ker(f(U))$

preshf $\mathrm{coker}(f)$

if $\mathcal{F} \hookrightarrow \mathcal{G}$ inclusion of AbPre ($\mathcal{F}(U) \subset \mathcal{G}(U)$ all U)

prequot $\mathcal{G}/\mathcal{F}$

Def sheaf im, ker, coker, quot = sheafification of presheaf im, ker, coker, quot.

Rem: preker = shf ker same for sheaves.

In particular, in either cat AbPre AbShv
can make sense of exact sequences

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$f = \ker g \quad g = \operatorname{coker} f$

$B/\operatorname{im} f \simeq C$