Divisors

X a scheme (Noetherian)

A Weil divisor on X is a formal integer linear combination of codimension 1 irreducible closed subsets

- [• Weil Divisor
- • Cartier Divisor
  things locally cut out by functions
- • Invertible sheaves

$\mathcal{L} \to X \rightsquigarrow c_1(\mathcal{L}) \in H^2(X, \mathbb{C})$

$H^{1,1}(X, \mathbb{C}) \subset H^2(X,\mathbb{C})$

$H^0(Z, \mathbb{C}) \xrightarrow{i_*} H^2(X,\mathbb{C})$ [1]

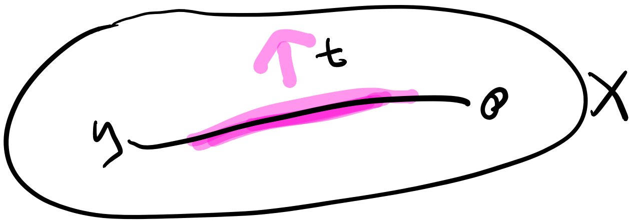
Z codim 2 in X

Condition ★: "RICO"

X is a Noeth. integral separated scheme which is regular in codim 1.

Reg. in codim 1 means: if $\mathfrak{P}$ is a codim 1 prime then $R_{\mathfrak{P}}$ is a regular local ring (of dimension 1)

$R_{\mathfrak{P}} = \mathcal{O}_{X,\mathfrak{P}}$



$t \in R_{\mathfrak{P}}$ uniformizer

for $r \in R_{\mathfrak{P}}$ maximally

$a \in F = \mathrm{frac}\, R_{\mathfrak{P}} = \mathrm{ff}(X)$

$a = ut^m$

Recall: A regular ring of dimension 1 is a discrete valuation ring!

In a dvr R, max. ideal $m = tR$ is principal
t "uniformizer" if $r \in R$, can uniquely write
$r = ut^n$ $u \in R^*$
and in fact for $a \in F = \text{frac}\, R$, can uniquely write
$a = vt^m$ $v \in R^*$

F^*/R^*
$[a] \geq [b]$ if $a = bc$, $c \in R \setminus 0$

$\boxed{\dfrac{R \setminus \{0\}}{R^*}}$

$\mathbb{Z}_{\geq 0}$

$F \setminus \{0\}/R^*$

get a map $\text{val}: F^* \longrightarrow \mathbb{Z}$ (homomorphism of Ab gps)
$vt^m \longmapsto m$

$\text{val}(ab) = \text{val}(a) + \text{val}(b)$
$\text{val}(a+b) \geq \min\{\text{val}(a), \text{val}(b)\}$

$\text{val}: F \longrightarrow \mathbb{Z} \cup \{\infty\}$
$0 \longmapsto \infty$

Def A prime Weil divisor on X is a closed integral subscheme of codim 1.

$\leftrightarrow$ (points w/ codim 1 closure) $\updownarrow$ (closed subsets irred, codim 1)

A Weil divisor is an elmt of the free Ab gp gen. by prime Weil divisors.
$\text{Div}\, X$

$D = \sum a_i Y_i$

let K = fn field of X

for $f \in K$, $Y \subset X$ prime Weil divisor, gen pt η_Y

have $K = \mathrm{frac}\, \mathcal{O}_{X,\eta_Y} \Rightarrow K^* \xrightarrow[v_Y]{\text{val}} \mathbb{Z}$

$\uparrow$ dvr

<u>Lem</u> for $f \in K^*$, $\{Y \subset X \mid v_Y(f) \neq 0\}$ is finite.

Pf: choose $\mathrm{Spec}\, A \subset X$ affine

$f = \frac{a}{b}$, $a, b \in A$

$\rightsquigarrow$ f regular in $\mathrm{Spec}\, A_b$ $D(b)$

$(X = \mathrm{Spec}\, \mathbb{Z})$

$K = \mathbb{Q}$

if $Y \subset X$, w/ $\eta_Y \in \mathrm{Spec}\, A_b$

then $f \in A_b \subset \mathcal{O}_{X,\eta_Y}$

$A_b = \mathcal{O}_X(\mathrm{Spec}\, A_b)$

$U \subset V$

$f \in \mathcal{O}_X(U) \leftarrow \mathcal{O}_X(V)$

$f \mapsto f$

$\Rightarrow v_Y(f) \geq 0$

So if $Y \subset X$, $v_Y(f) < 0 \Rightarrow$

$\eta_Y \notin \mathrm{Spec}\, A_b$

$\eta_Y \in \underbrace{X \setminus \mathrm{Spec}\, A_b}_{\text{closed}}$

union of finitely many irred components each of which

21

$v_3(30)$ $\quad \mathbb{Z}_{(3)} = \left\{ \frac{a}{b} \mid 3 \nmid b \right\}$

$21 = \frac{42}{2} = 6^1 \cdot \left(\frac{7}{2}\right)$

unit

has codim ≥ 1

$\Rightarrow \# \{ y \subset X \mid v_y(f) < 0 \}$ is finite.

$\Rightarrow \# \{ y \subset X \mid v_y(f^{-1}) < 0 \}$ is finite

$$v_y(f^{-1}) = -v_y(f) \qquad \square.$$

Def If $f \in K^*$ we define $\operatorname{div}(f) = (f)$ is defined as $(f) = \sum_y v_y(f) \cdot y$

Def $D \in \operatorname{Div} X$ is <u>principal</u> if $D = (f)$ some f.

Note: $K^* \longrightarrow \operatorname{Div} X$ Ab. gp hom.
$f \longmapsto (f)$

$\Rightarrow$ Principal divisors are a subgp of $\operatorname{Div} X$
$\operatorname{Prin} X$

Def $Cl(X) = \dfrac{\operatorname{Div} X}{\operatorname{Prin} X}$

Ex: if F a # field (a finite ext of $\mathbb{Q}$)
$\mathcal{O}_F$ = ring of ints (int closure of $\mathbb{Z}$ in F)
that's the notation

then $Cl(F) = Cl(\operatorname{Spec} \mathcal{O}_F)$

defined in #thy ↑ ↑ above.

Prop: If A is a Noeth domain then A is a UFD

$\iff$ A is integrally closed & $Cl(\operatorname{Spec} A) = 0$.

Cartier Divisors

Cartier Divisors $\underset{\text{morally}}{=}$ locally principal divisors

loc. principal divisor would be constructed as:

a cover $\{U_i\}$ of X

rat'l functions f_i on U_i "$(f_i)|_{U_i}$"

s.t. $f_i|_{U_i \cap U_j}$ & $f_j|_{U_i \cap U_j}$ have same 0's & poles.

i.e. $f_i f_j^{-1}|_{U_i \cap U_j}$

in $\mathcal{O}_X(U_i \cap U_j)^*$

In general, need to define a sheaf $\underbrace{K_X}$ $(\mathcal{O}_X)$

<u>Def</u> $\operatorname{CaDiv} X = \Gamma(X, K_X^*/\mathcal{O}_X^*)$ $K_X(U) = \mathcal{O}_X(U)[\operatorname{Reg}^{-1}]$

$K_X = \widetilde{K_X}^+$

all nonzero divisors

Alt. Def

$$\mathrm{CaDiv}\,X = \left\{ \begin{array}{l} (U_i, f_i) \text{ some open cover } f_i \in K_X(U_i)^* \\ \text{s.t. } \quad f_i f_j^{-1}\big|_{U_i \cap U_j} \in \mathcal{O}_X(U_i \cap U_j)^* \end{array} \right\} \Big/ \sim$$

$(U_i, f_i) \sim (V_j, g_j)$

if $\exists$ refinement $\{W_k\}$ of these covers

$$(W_k, f_{i(k)}) \qquad (W_k, g_{j(k)})$$

$$f_{i(k)} = \mathrm{res}^{U_{i(k)}}_{W_k} f_{i(k)} \qquad f_{i(k)} = u_k\, g_{j(k)} \qquad u_k \in \mathcal{O}_X(W_k)^*$$

$W_k \qquad U_i$

$$W_1 \to U_1, \qquad W_2 \to U_1, \qquad U_2$$

$i(2) = 1$