Quick Cartier Summary

frac R = total ring of fractions
= $R[S^{-1}]$ S = regular elements

Practical definition

A Cartier divisor is given by [a cover U_i and $f_i \in (\mathrm{frac}\, \mathcal{O}_X(U_i))^*$ for each i

represent by $[U_i, f_i]_{i \in I}$

such that $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^* \subset (\mathrm{frac}\, \mathcal{O}_X(U_i \cap U_j))^*$

$[U_i, f_i] \sim [U_i, g_i]$ if $f_i/g_i \in \mathcal{O}_X(U_i)^*$

and if $\{V_{ij}\}_j$ cover U_i then $[U_i, f_i] \sim [V_{ij}, f_i|_{V_{ij}}]$

we let (U_i, f_i) denote the equiv. classes in eq. rel.

gen. by these.] (U_i, f_i)

$$\mathrm{CaDiv}(X) = \{ (U_i, f_i) \} \text{ eq. classes as above.}$$

Alternately: $\mathrm{CaDiv}(X) = \Gamma(X, \mathcal{K}_X^* / \mathcal{O}_X^*)$

Remark: these form a gp, induced by

$$(U_i, f_i) + (U_i, g_i) = (U_i, f_i g_i)$$

(more generally, make a common refinement)

Def $\mathrm{CaPrin}(X) = \{ (U_i, f|_{U_i}) \mid U_i \text{ cover}, f \in \mathcal{K}_X^*(X) \}$

$$\mathrm{CaCl}(X) = \mathrm{CaDiv}(X) \big/ \mathrm{CaPrin}(X)$$

Suppose X is ☆ (Noeth, sep, RICO) then we have a well defined map

$$\mathrm{CaDiv}\,X \longrightarrow \mathrm{Div}\,X \quad \text{(Weil divisors)}$$

$$D = (U_i, f_i) \longmapsto \sum_{\substack{Z \text{ irred} \\ \text{codim } 1 \\ \text{closed}}} v_Z(D) \cdot Z$$

$v_Z(D) = v_Z(f_i)$ for i s.t. $\eta_Z \in U_i$

Z codim 1 irred

η_Z = gen. pt, $\mathcal{O}_{X,\eta_Z}$ disc. val ring.

note if $\eta_Z \in U_i \cap U_j$

$v_Z(f_i) = v_Z(f_j)$

since $f_i / f_j \in \mathcal{O}_X(U_i \cap U_j)^*$

$\Rightarrow f_i / f_j \in \mathcal{O}_{X,\eta_Z}^* =$ things w/ value 0

<u>Prop (6.11)</u> If X is separated, integral, locally factorial ($\Rightarrow$ RICO, ☆)

then $\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$

is an isomorphism of groups and

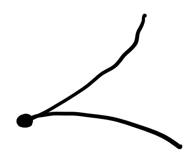
induces an iso $\mathrm{CaPrin}(X) \to \mathrm{Prin}(X)$

$$\mathrm{Ca\,Div}(X) \big/ \mathrm{Ca\,Prin}(X) \simeq \mathrm{Ca\,Cl}(X)$$

$\wr$ under these hypotheses.

$$\mathrm{Div}\,X \big/ \mathrm{Prin}\,X \simeq \mathrm{Cl}(X)$$

Non-example: $X = \mathrm{Spec}\ \dfrac{k[x,y]}{y^2-x^3}$



$z \longleftrightarrow (x,y)$

then z is a Weil divisor

$1 \cdot z$ not Cartier.

idea: z is not (locally) principal.

$\mathcal{O}_{X,z}$ max'l ideal, not cut out by a single fcn.

Invertible Sheaves

Observation: if $\mathcal{L}, \mathcal{M}$ are locally sheaves of $\mathcal{O}_X$-mods on a scheme X, of ranks n, m then

- $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is loc. free, rk nm
- $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is loc. free, rk n

and if $n=1$, the map

- $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \longrightarrow \mathcal{O}_X$ is an isomorphism.

loc. presheaf level

$$\sum s_i \otimes f_i \longmapsto \sum f_i(s_i)$$

and on stalks $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X$ mult.

$$\mathcal{O}_X^n \otimes_{\mathcal{O}_X} \mathcal{O}_X^n \to \mathcal{O}_X$$

__Def__ Pic X = {isom. classes of inv. sheaves}, $\otimes$

an Abelian gp!

__Amazing fact__ (conclusion today)

every invertible sheaf $\mathcal{L}$ is $\cong$ to a subsheaf of $\mathcal{K}_X$

if X is integral

(think fractional ideals for Dedekind domains)

R dd. $K = \operatorname{frac}(R)$

if M is proj. rk 1 /R

$\exists\, M \hookrightarrow K$ as R-mods.

__Def__ a fractional invertible sheaf is an invertible subsheaf of $\mathcal{K}_X$.

Given a Cartier divisor $D = (U_i, f_i)$, can define a fractional invertible sheaf via:

$$\mathcal{L}(D)\Big|_{U_i} = f_i^{-1}\mathcal{O}_{U_i}$$

an $\mathcal{O}_X(V)$-module

i.e. $\mathcal{L}(D)(V) = f_i^{-1}\mathcal{O}_X(V) \subset \mathcal{K}_X(V)$

$V \subset U_i$ $\qquad f_i \in \mathcal{K}_X(U_i)^*$

well defined since for $V \subset U_i \cap U_j$

$f_i, f_j \in \mathcal{K}_X(U_i \cap U_j)^*$

we have $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^*$

so $f_i^{-1}\mathcal{O}_X(V) = u \cdot f_i^{-1}\mathcal{O}_X(V) = f_j^{-1}\mathcal{O}_X(V)$

$f_i/f_j = u \in \mathcal{O}_X(U)^*$

Ex: if $g_i = f_i^{-1} \in \mathcal{O}_X(U_i)$

then $\mathcal{L}(D)(U_i) = g_i\mathcal{O}_X(U_i)$ = ideal gen by g_i

= fcns which vanish along $Z(g_i)$

$\mathcal{L}(D)$ = sheaf of ideals which cuts out the Cartier divisor "(g_i)"

Prop (6.13) X is any scheme

$$\mathrm{CaDiv}(X) \longrightarrow \mathrm{FInv}(X) \quad = \text{fractional invertible sheaves}$$
$$D \longmapsto \mathcal{L}(D)$$

is bijective
and induces a group homomorphism to $\mathrm{Pic}\, X$

$$\mathcal{L}(D+D') \simeq \mathcal{L}(D) \otimes_{\mathcal{O}_X} \mathcal{L}(D')$$

and $D \sim D'$ $\iff$ $\mathcal{L}(D) \simeq \mathcal{L}(D')$ as $\mathcal{O}_X$-mods.

via principal

$$\Longrightarrow \quad \mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}(X)$$

Prop 6.15 If X is integral, this is an isomorphism.

Cor If X is integral, separated, Noeth, locally factorial
then $\mathrm{Cl}(X) \simeq \mathrm{CaCl}(X) \simeq \mathrm{Pic}\, X$

Def A Cartier divisor $D \leftrightarrow (U_i, f_i)$ is **effective** if $f_i \in \mathcal{O}_X(U_i)$

Def A Weil divisor $D = \sum n_i Z_i$ is effective if each $n_i \geq 0$.

If a Cartier divisor $D = (U_i, f_i)$ is effective,
we define the assoc. subscheme of D
is the closed subscheme w/ sheaf of ideals
$\mathcal{I}_D$ where $\mathcal{I}_D|_{U_i} = f_i \mathcal{O}_X|_{U_i}$

Conversely given a closed subscheme $Y \subset X$ which
is locally principal (i.e. $\mathcal{I}_Y|_{U_i} = g_i \mathcal{O}_X|_{U_i}$ some open cover U_i)

then $\mathcal{I}_Y$ is an invertible subsheaf of $\mathcal{K}_X$
(i.e. a fractional invertible sheaf)
and so is given by a Cartier divisor.

If Cartier divisor $D \longleftrightarrow Y \subset X$ loc. principal closed subscheme

then $\mathcal{L}(-D) \simeq \mathcal{I}_Y$

<u>Remark</u>: We "showed"
if $X = \mathbb{P}^n_k$ then for D a Weil divisor, $D \sim dH$
H a hyperplane.

$\Rightarrow Cl(\mathbb{P}^n_k) \simeq \mathbb{Z}$
$Pic(\mathbb{P}^n_k) \simeq \mathbb{Z}$ generated by $\mathcal{O}(H)$

and can see $\mathcal{L}_H \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$

$$0 \to x_0 k[x_0, \ldots, x_n] \longrightarrow k[x_0, \ldots, x_n] \longrightarrow k[x_0, \ldots, x_n]/x_0 \to 0$$

$$x_0 k[x_0, \ldots, x_n] = I_H \qquad H = Z(x_0)$$

$$\mathcal{L}_H = \widetilde{I}_H \cong \mathcal{O}_{\mathbb{P}^n_k}(1) \qquad I_H \simeq k[x_0, \ldots, x_n][1]$$