(Section 7)

Morphisms to projective space & globally generated line bundles

Def A coherent sheaf $\mathcal{F}$ on a scheme X is <u>globally generated</u> if $\exists$ sections $s_1, \ldots, s_n$ (in general maybe not finite) s.t. $\forall x \in X$ images of $s_1, \ldots, s_n$ generate $\mathcal{F}_x$.

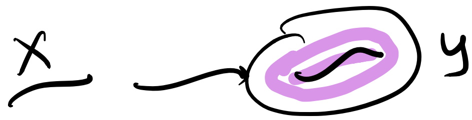
First Surprise: globally generated line bundles always give (and come from) maps to projective space.

More precisely: X is a scheme over A (comm ring) recall, we have defined the inv-sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n_A$

(if $\mathbb{P}^n_A = \text{Proj}\, A[x_0, \ldots, x_n]$, then $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n_A, \mathcal{O}(1))$ and $\mathcal{O}(1)$ is globally generated by these

and
Thm 7.1
- if $\varphi : X \longrightarrow \mathbb{P}^n_A$ is any morphism, then $\varphi^* \mathcal{O}(1)$ is globally generated.
- if $\mathcal{L}$ is any globally generated line bundle on X, $\exists\, \varphi : X \to \mathbb{P}^n_A$ s.t. $\varphi^* \mathcal{O}(1) \simeq \mathcal{L}$.

First part: recall from last time if $f: X \to Y$ any morphism, $\mathcal{F}$ q.coh on Y, get an induced map $f^*: \Gamma(\mathcal{F}) \to \Gamma(f^*\mathcal{F})$



in particular, given $\varphi: X \to \mathbb{P}^n_A$, set $s_i = \varphi^*(x_i)$

and can check that the gen. $\varphi^*\mathcal{O}(1)$

$$\mathcal{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^n}(1)$$

on U, $a_i \in \mathcal{O}_{\mathbb{P}^n}(U)$

$$(a_0, \dots, a_n) \longmapsto \sum a_i x_i|_U$$

glob. gen $\Rightarrow \psi_x$ is surj (i.e. ψ is surj map of sheaves)

$$\begin{tikzcd} X \arrow[r] & \mathbb{P}^n_A \\ x \arrow[r, mapsto] & p \\ \mathcal{O}_{X,x} & \mathcal{O}_{\mathbb{P}^n_A, p} \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} \varphi^*\mathcal{O}_{\mathbb{P}^n}^{n+1} \arrow[r, "\varphi^*\psi"] \arrow[d, equal] & \varphi^*\mathcal{O}(1) \\ \mathcal{O}_X^{n+1} \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}_{X,x}^{n+1} \arrow[r] \arrow[d, equal] & \varphi^*\mathcal{O}(1)_x \arrow[d, equal] \\ \mathcal{O}_{\mathbb{P}^n,p}^{n+1} \otimes_{\mathcal{O}_{\mathbb{P}^n,p}} \mathcal{O}_{X,x} \arrow[r, "\psi_x \otimes \mathcal{O}_{X,x}"'] & \mathcal{O}(1) \otimes_{\mathcal{O}_{\mathbb{P}^n_A,p}} \mathcal{O}_{X,x} \end{tikzcd}$$

Quick heuristic

Given an inv. sheaf $\mathcal{L}$ on X, globally generated by $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$

Define a map $\varphi : X \longrightarrow \mathbb{P}^n_A$ as follows:

$$x \longmapsto \text{"}[s_0(x) ; s_1(x) ; \dots ; s_n(x)]\text{"}$$

but ~~wait~~ there aren't #'s.

note that $\dfrac{s_i(x)}{s_j(x)}$ is well defined (i.e. doesn't depend on $\mathcal{L} \simeq \mathcal{O}_X$ locally) when $s_j(x) \neq 0$

Poetry, ~~not math~~. (i.e. varieties over the complex #s)

slightly better:

for each i, let $U_i \subset X$ be locus where $s_i \neq 0$ i.e. $U_i = \{x \in X \mid s_i|_x \notin m_x \mathcal{L}_x\}$

$s_i|_x \in \mathcal{L}_x$

and then for $V \subset U_i$ s.t. $\mathcal{L}|_V \simeq \mathcal{O}_X|_V$

choose an isom. $\psi_i : \mathcal{L}|_V \xrightarrow{\sim} \mathcal{O}_X|_V$

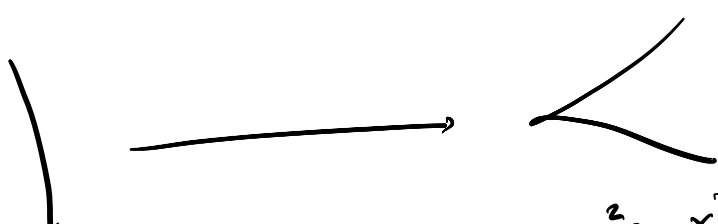
define
$$\begin{tikzcd} V \arrow[r] \arrow[dr] & \mathbb{P}^n_A \\ & \mathbb{A}^{n,i}_A \arrow[u] \end{tikzcd}$$

$$\mathbb{A}^{n,i}_A \longleftarrow V$$

$$A[y_1, \dots, y_n]$$

$$A\left[\frac{x_1}{x_i}, \frac{x_2}{x_i}, \dots, \frac{x_n}{x_i}\right] \longrightarrow \mathcal{O}_X(V)$$

$$\frac{x_j}{x_i} \longmapsto \frac{\psi(s_j)}{\psi(s_i)}$$



$\subset \mathbb{P}^2$

$y^2 z = x^3$

$\mathbb{P}^1$

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "id"] \arrow[dr] & \mathbb{P}^1 \\ \mathcal{O}(1) & \mathbb{P}^2 \end{tikzcd}$$

$\left(\mathcal{L} + \text{sections } s_0, \ldots, s_n \right) \longleftrightarrow$ maps to proj space.

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when is a map to proj. space an embedding?

Prop if $X \xrightarrow{\varphi} \mathbb{P}^n_A$ is a morphism given by $\mathcal{L}, s_0, \ldots, s_n$ sections as above,

then φ is a closed embedding if and only if

$X_i = \varphi^{-1}(\mathbb{A}^{n,i}_A)$ is affine, and

$A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i})$

is surjective.

Prop If $A = k = \bar{k}$ an alg. closed field, then as above have a closed immersion if and only if

"s_i's separate pts & tangent vectors"

i.e. if $V = \langle s_0, \ldots, s_n \rangle$ in $\Gamma(X, \mathcal{L})$ (a k-vector space)

and $P, Q \in X$, then $\exists\ s \in V$ s.t. $s_P \notin m_P \mathcal{L}$, $s_Q \in m_Q \mathcal{L}$.

- for all P, $\{ s \in V \mid s_P \in m_P \mathcal{L} \}$ spans $m_P\mathcal{L}/m_P^2\mathcal{L}$

Def An invertible sheaf on a Noeth scheme X is <u>ample</u> if for every coherent sheaf $\mathcal{F}$ on X $\exists\ n_0 > 0$ integer s.t. $\forall\ n \geq n_0$, $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{F}$ is globally generated.

<u>Note</u>: any coherent sheaf on an affine scheme is glob. gen.
$\Rightarrow$ every inv-sheaf is ample.

Def An invertible sheaf $\mathcal{L}$ is very ample if $\exists\ \varphi: X \to \mathbb{P}^n_A$ closed embedding w/ $\mathcal{L} \cong \varphi^* \mathcal{O}(1)$

<u>Thm (Serre)</u> very ample $\Rightarrow$ ample

<u>Thm 7.6</u> If X is finite type over a Noeth ring A, and $\mathcal{L}$ is an invertible sheaf on X then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes m}$ is very ample for some $m > 0$.

A tiny bit of the proof

choose $x \in X$, want to find some n,
section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ s.t. $X_s = \{p \in X \mid s_p \notin m_p \mathcal{L}\}$
and $x \in X_s$ (i.e. X_s's will cover)

choose $U \ni x$ s.t. $\mathcal{L}|_U \simeq \mathcal{O}_X|_U$

suppose $Y = X \setminus U$.

going to find a section s which vanishes all along Y, not at x. ($\Rightarrow \underset{\substack{\Downarrow \\ x}}{X_s} \subset U$ affine)

$\mathcal{I}_Y \hookrightarrow \mathcal{O}_X$

" "stuff that vanishes along Y"

$\mathcal{I}_Y \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \hookrightarrow \mathcal{L}^{\otimes n}$

"stuff in $\mathcal{L}^{\otimes n}$ that vanishes along Y"

if $n \gg 0$, $\mathcal{I}_Y \otimes \mathcal{L}^{\otimes n}$ is glob. gen.

$\Rightarrow \exists\ \tilde{s} \in \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^{\otimes n})$
s.t. $\tilde{s} \notin m_x (\mathcal{I}_Y \otimes \mathcal{L}^{\otimes n})_x$

$\hphantom{s.t. \tilde{s} \notin m_x (\mathcal{I}_Y} \cong \mathcal{L}_x^{\otimes n}$

consider image s of $\tilde{s}$ in $\mathcal{L}^{\otimes n}$

$(\mathcal{I}_Y)_x = \mathcal{O}_{X,x}$

via this isom,

$s \notin m_x \mathcal{L}^{\otimes n}$