Philosophical perspective on ampleness intuition.

$\mathcal{L}$ ample means can find global sections of $\mathcal{L}^{\otimes n}$ $n \gg 0$ with specific prescribed behavior.

ex: X Noeth scheme, $Z \subset X$ closed reduced $p \in X \setminus Z$ then we can find $n \gg 0$, $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $s|_Z = 0$ and $s(p) \neq 0$

what does this even mean? (translating $s|_Z = 0$)

1. $i: Z \hookrightarrow X$ closed embedding $i^*\mathcal{L}$ and consider $i^*s \in \Gamma(Z, i^*\mathcal{L})$ statement is $i^*s = 0$

2. for $q \in Z$, $s_q \in m_q (\mathcal{L}^{\otimes n})_q$

3. $\exists\, U_i \subset X$ open affine cover s.t. $\mathcal{L}^{\otimes n}|_{U_i} \xrightarrow[\varphi_i]{\sim} \mathcal{O}_{U_i}$ and $U_i \simeq \operatorname{Spec} A_i$, $I_i \trianglelefteq A_i$ ideal of $Z \cap U_i$ then $\varphi_i(U_i)(s|_{U_i}) \in I_i$

4. $\exists\, U_i \subset X$ open cover s.t. $\mathcal{L}^{\otimes n}|_{U_i} \xrightarrow[\varphi_i]{\sim} \mathcal{O}_{U_i}$ s.t. $\varphi_i(U_i)(s|_{U_i}) \in \mathcal{I}_Z(U_i)$ ← ideal sheaf of U_i

5 · If Z irred w/ gen. pt η_Z, also same as

$$(s)_{\eta_Z} \in m_{\eta_Z} \cdot (\mathcal{L}^{\otimes n})_{\eta_Z}$$

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pf is: $\mathcal{I}_Z$ ideal shf $\leftrightarrow$ fns which vanish along Z

$$\mathcal{I}_Z \subset \mathcal{O}_X \quad \text{and} \quad (\mathcal{I}_Z)_p = \mathcal{O}_{X,p}$$

a section $s \in \Gamma(X, \mathcal{I}_Z) \subset \Gamma(X, \mathcal{O}_X)$

corresp. to my fns which vanish along Z

and want to find one w/ $s_p \neq 0$

[if X is proper scheme / $k = \bar{k}$ then $\Gamma(X, \mathcal{O}_X) = k$]

but if $\mathcal{L}$ ample $\Rightarrow$ $\mathcal{I}_X \otimes \mathcal{L}^{\otimes n}$ glob. gen

some $n \gg 0$

$\Rightarrow (\mathcal{I}_X \otimes \mathcal{L}^{\otimes n})_p$ is gen. by global sections.

$\Rightarrow \exists\, s \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n})$ s.t. $s|_p \notin m_p (\mathcal{I}_X \otimes \mathcal{L}^{\otimes n})_p$

$\Rightarrow s \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) \subset \Gamma(X, \mathcal{L}^{\otimes n})$

$\mathcal{I}_Z \subset \mathcal{O}_X \quad \mathcal{I}_Z \otimes \mathcal{L}^{an} \subset \mathcal{L}^{an}$

($\Longleftarrow$ via Cartier divisors
can find rat'l fcn w/ prescribed behavior
at various pts, closed subschemes etc
as long as we allow poles in some
fixed places)

<u>Recall:</u>
proj varieties are the zero locus of sets
of hom. polynomials.
" global sections of $\mathcal{O}(n)$ various $n>0$
$\mathcal{O}_{\mathbb{P}^n}(1)$ lin polys $\mathcal{O}(2)$ quad.
in $k[x_0, \dots, x_n]$

more generally, if $\mathcal{F}$ some q-coh sheaf on X
can consider its vanishing locus
$s \in \Gamma(X, \mathcal{F}) \quad Z(s) = \{ p \in X \mid s_p \in m_p \mathcal{F}_p \}$

But this is too coarse for today.

if $\mathcal{F} \rightsquigarrow \mathcal{L}$ inv. sheaf

then for $s \in \Gamma(X, \mathcal{L})$ can define a more refined

vanishing locus $(s)_0$ = effective Cartier divisor.

<u>Def</u> $(s)_0 = \{(U_i, f_i)\}$ via:

let U_i be cover s.t. $\exists$ isoms $\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$

and we set $f_i = \varphi_i(U_i)(s|_{U_i})$

(defined when s regular

e.g. X integral $s \not\equiv 0$)

note $(s)_0$ is effective since $f_i \in \mathcal{O}_X(U_i) \subset K_X(U_i)$

D_0 Cartier $\rightsquigarrow \mathcal{L}(D_0)$ line bundle

$\{(U_i, f_i)\} \rightsquigarrow \mathcal{L}(D_0) \subset K$ where

$\mathcal{L}(D_0) = \mathcal{L}$

$\mathcal{L}(D_0)|_{U_i} = f_i^{-1}\mathcal{O}_{U_i} \cong \mathcal{O}_{U_i}$

s

$$K \xrightarrow{\cdot f_i} K$$

$\mathcal{L}$

$(s)_0 \leftsquigarrow s \in \Gamma(X, \mathcal{L})$

divisor.

<u>Examine this:</u> given X integral F = function field of X

suppose given $D_0 = \{(U_i, f_i)\}$. $K(U) = F$

Cartier, $f_i \in K(U_i) = F$

$s \in \Gamma(X, \mathcal{L}(D_0))$ examine $(s)_0$

$\cap$

$\Gamma(X, K_X) = F$

$s \in F$ consider $s|_{U_i} \in \mathcal{L}(D_0)(U_i) \xrightarrow[\varphi_i]{\sim} \mathcal{O}_X(U_i)$

$\cap$

$s \in F \to f_i s$

$\mathcal{L}(D_0)|_{U_i} = f_i^{-1} \mathcal{O}_{U_i}$

$s \cdot f_i = g_i \in \mathcal{O}(U_i)$

$s \in F \rightsquigarrow (s)_0 = \{(U_i, g_i)\}$

$g_i = \varphi_i(s) = f_i s$

notice therefore by construction

$$\{(U_i, g_i)\} = \underbrace{\{(U_i, f_i)\}}_{D_0} + \{(U_i, s)\}$$

principal divisor.

$$\|$$

i.e. $(s)_0$ is an effective divisor, linearly equiv. to D_0.

Conversely if $\{(U_i, g_i)\}$ effective Cartier & lin. equiv to $\{(U_i, f_i)\}$

then i.e. $\{(U_i, g_i)\} = \underbrace{\{(U_i, f_i)\}}_{D_0} + \{(U_i, s)\}$

and so $s f_i = g_i \in \mathcal{O}_X(U_i) \implies s \in \Gamma(X, \mathcal{L}(D_0)) \subset \Gamma(X, K_X)$

$$\Updownarrow$$

$s = g_i f_i^{-1} \in \mathcal{O}_X(U_i) f_i^{-1} = \mathcal{L}(D_0)(U_i)$ $\cap K(U_i)$

Surjective map

$$\Gamma(X, \mathcal{L}(D_0)) \setminus \{0\} \longrightarrow \left\{ \begin{array}{c} \text{effective Cartier divisors,} \\ \text{linearly equiv. to } D_0 \end{array} \right\}$$

note: if $\underset{\Downarrow}{s, s'} \longmapsto$ same effective divisor

$s \rightsquigarrow (s_i, U_i)$ $\qquad$ $s' \rightsquigarrow (s_i', U_i)$

$\quad\parallel$ $\qquad\qquad\qquad$ $\parallel$

$s_i f_i^{-1}$ $\qquad\qquad$ $s_i' f_i^{-1}$

same Cartier divisor $\qquad$ $s_i' = v_i s_i$

v_i unit.

$$\Rightarrow \frac{s}{s'}\Big|_{U_i} = \frac{s_i}{s_i'}\Big|_{U_i} = v_i\Big|_{U_i} \text{ unit}$$

$$\Rightarrow \lambda = \frac{s}{s'} \qquad \lambda s' = s$$

λ is in $\mathcal{O}_X(X)^*$

$\qquad\qquad\qquad\parallel$

in case $k = \bar{k}$, X an integral proper $\bar{k}$-scheme $\Gamma(X, \mathcal{O}_X)^*$

we have $\lambda \in k^*$

so get a bijection

$$\mathbb{P}(\Gamma(X, \mathcal{L}(D_0))) = \Gamma(X, \mathcal{L}(D_0)) \big/ k^* = \left\{ \begin{array}{l} \text{effective cartier divisors} \\ D, \text{ linearly equiv to } D_0 \end{array} \right\}$$

Def $|D_0| = \{$ effective divisors, linearly equiv. to $D_0 \}$

$$= \mathbb{P}(\Gamma(X, \mathcal{L}(D_0)))$$

"full linear system of D_0"

Def A linear system is a linear subspace $\delta \subset \mathbb{P}(\Gamma(X, \mathcal{L}(D_0)))$ some D_0 Cartier

Def if $P \in X$ we say P is a base point of a linear system δ if $P \in \text{support}(D)$ all $D \in \delta$

Lemma δ base point free $\iff$ globally generated

i.e. bpf $\iff$ can use δ to define a morphism to proj. space.

(i.e. elements in v. space assoc. to $\delta \rightsquigarrow$ global sections which generate $\mathcal{L}(D_0)$)

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choose $s_0, \dots, s_n$ gen. underlying v. space for δ.

$$\delta \subset \Gamma(X, \mathcal{L}(D_0)) \subset \Gamma(X, K) = F$$

and if $D_0 = \{(U_i, f_i)\}$

$s_j|_{U_i} = g_{ij} f_i^{-1} \quad g_{ij} \in \mathcal{O}_X(U_i) \setminus \{0\}$

"map defined by δ"

$p \longmapsto [s_0(p) ; \ldots ; s_n(p)] \quad s_i \in F$

on U_i can rewrite this as

$[s_0(p) ; \ldots ; s_n(p)] = [(g_{i0} f_i^{-1})(p) ; \ldots ; (g_{in} f_i^{-1})(p)]$

$= [g_{i0}(p) ; \ldots ; g_{in}(p)]$

small problem: what if these all vanish at some p?

$\Longleftrightarrow$

all effective divisors described by δ contain P.

i.e. $\Leftrightarrow p \in$ base locus of δ

Punchline:

$\mathcal{L}(D_0)$ defines a map

$X \supset U \longrightarrow \mathbb{P}^n$

complement of base locus.