Plan for today:

Ringed spaces, Locally ringed spaces, Spec, Schemes

Ringed Spaces

Def A ringed space is a top space X, together with a sheaf of rings $\mathcal{O}_X$. $(X, \mathcal{O}_X)$

$\mathcal{O}_X$ = "good functions on X" (cont, smooth, analytic, polynomial)

A morphism of ringed spaces

$$(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

$f : X \longrightarrow Y$ cont map

and a "pullback" map

$$g \in \mathcal{O}_Y(U) \rightsquigarrow f^{\#}(g) \in \mathcal{O}_X(f^{-1}U) = f_*\mathcal{O}_X(U)$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed] & Y \arrow[d, "g"] \\ & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} f^{-1}U \arrow[r, "f"] \arrow[dr, dashed] & U \arrow[d, "g"] \\ & \mathbb{C} \end{tikzcd}$$

$$f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$$

alternately, if $V \subset X$, $U \supset f(V)$

$$\lim_{\overrightarrow{U \supset f(V)}} \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(V)$$

$$\parallel$$

$$f^{-1}\mathcal{O}_Y(V)$$

" $f^{\#} : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ "

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, dashed] & U \arrow[d] \\ & \mathbb{C} \end{tikzcd}$$

<u>Def</u>: A morphism of ringed spaces

$$(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is a pair $(f, f^{\#})$ where

$f : X \longrightarrow Y$ cont &

$f^{\#} : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$

Locally ringed spaces

"Desires" $(X, \mathcal{O}_X)$ ringed space

- if $p \in X$ and $f \in \mathcal{O}_X(U)$ w/ $p \in U$ want an "evaluation" $f(p)$

$p \in V \subset U$
$$\begin{array}{ccc} \mathcal{O}_X(U) & \to & k(p) \\ \downarrow & \nearrow & \\ \mathcal{O}_X(V) & & \end{array}$$

- $f \in \mathcal{O}_X(U)$ then $\{x \in U \mid f(x) = 0\}$ closed in U
- if $f \in \mathcal{O}_X(U)$ and $p \in U$, $f(p) \neq 0$ then $\exists_{p \in} V \subset U$ s.t. $f(x) \neq 0$ all $x \in V$, and we'll want $1/f$ to be defined on V $\uparrow \mathcal{O}_X(V)$

- $\Rightarrow$ values $f(p)$ should lie in fields. $\underbrace{f(p) \in k(p)}_{\text{field}}$ depend on pt p.

- if $f(p) \neq 0$ then f is invertible in some nbhd of p
  $\Rightarrow \left(f(p) \neq 0 \Rightarrow f_p \in \mathcal{O}^*_{X,p} \right)$

- Let $m_p = \{ g \in \mathcal{O}_{X,p} \mid g(p) = 0 \}$
  above says: if $g \notin m_p \iff g \in \mathcal{O}^*_{X,p}$
  in other words: $\mathcal{O}_{X,p}$ should be a local ring w/ max'l ideal m_p.

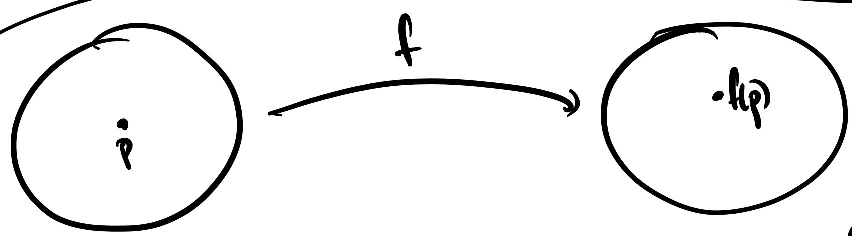
Def A locally ringed space $(X, \mathcal{O}_X)$ is a ringed space s.t. all stalks $\mathcal{O}_{X,p}$ are local rings for $p \in X$.

notation: $m_p \triangleleft \mathcal{O}_{X,p}$ max'l ideal.

"evaluation" $g \in \mathcal{O}_X(U)$ $\quad p \in U$

$$\begin{tikzcd}
g \in \mathcal{O}_X(U) \arrow[d] & \\
g_p \in \mathcal{O}_{X,p} \arrow[r] \arrow[dr] & \kappa(p) \arrow[d, phantom, "\cong"] \\
& \mathcal{O}_{X,p}/m_p
\end{tikzcd}$$

Def A local map of locally ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of ringed spaces



such that induced map: $f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$

$f_p^\#(m_{f(p)}) \subset m_p$

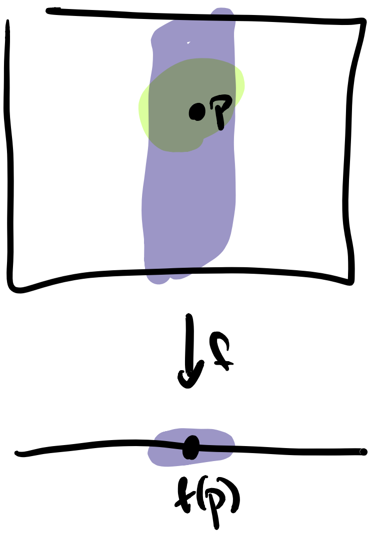
$$f^{\#}(U) : \mathcal{O}_Y(U) \longrightarrow f_*\mathcal{O}_X(U)$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathcal{O}_X(f^{-1}(U))$$

limit

$$(f^{\#})_p : \mathcal{O}_{Y,f(p)} \longrightarrow (f_*\mathcal{O}_X)_p$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \boxed{\varinjlim_{U \ni f(p)} \mathcal{O}_X(f^{-1}U)}$$

$f^{-1}U \ni p$

$$\boxed{\varinjlim_{V \ni p} \mathcal{O}_X(V) = \mathcal{O}_{X,p}}$$



Schemes

Basic inspiration: all commutative rings are functions on spaces.

max'l ideals $\rightsquigarrow$ give points in this space.

evaluation at a point: $R \longrightarrow k(p)$ ← field

functions on space $\mathcal{O}_X(U)$

homomorphism

every eval. to every field should be ok.

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & K \\ & R/I \arrow[u, hook] \end{tikzcd}$$

I prime.
and $K \supset \operatorname{frac} R/I$

pts $\longleftrightarrow$ prime ideals $I = P$

$\kappa(P) = \operatorname{frac} R/P$

$$R \longrightarrow R/P \longrightarrow \operatorname{frac} R/P$$

Spec R = locally ringed space associated to R.

$\operatorname{Spec} R = (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$
as a set, $\operatorname{Spec} R = \{P \subseteq R \text{ prime}\}$

- $\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) = R$
- If $f \in R$, $D_f = \{p \in \operatorname{Spec} R \mid f(p) \neq 0\}$
  are open and $D_f = \operatorname{Spec} R[f^{-1}]$
  $\mathcal{O}_{\operatorname{Spec} R}(D_f) = \mathcal{O}_{D_f}(D_f) = R[f^{-1}]$

<u>that's it</u>.

Topology: D_f are a basis.
$D_f \cap D_g$

$D_f \cap D_g$

$$D_f = \{p \in \operatorname{Spec} R \mid f(p) \neq 0\} = \{P \triangleleft R \text{ prime} \mid \bar{f} \in \operatorname{Frac}(R/P)\}$$
$$ \qquad \qquad \qquad \neq 0$$
$$= \{P \triangleleft R \text{ prime} \mid f \notin P\}$$

$$D_f \cap D_g = \{P \in \operatorname{Spec} R \mid f \notin P \text{ and } g \notin P\}$$
$$\overset{\shortparallel}{D_{fg}} \qquad = \{P \in \operatorname{Spec} R \mid fg \notin P\}$$

$$Z(f) = D_f^c = \{P \in \operatorname{Spec} R \mid f \in P\}$$

general closed set

$$\bigcap_{i \in I} Z(f_i) = \{P \in \operatorname{Spec} R \mid f_i \in P \text{ all } i\}$$
$$\langle f_i \rangle \subset P$$

$$Z(I) = \{P \in \operatorname{Spec} R \mid I \subset P\}$$ closed sets

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Abstract nonsense aside.

What we have so far

Spec R as a top space. closed sets $Z(I)$

only defined $\mathcal{O}_{\operatorname{Spec} R}(D_f) = R_f = R[f^{-1}]$

$D_f \supset D_g$

$$\begin{tikzcd} R_f \arrow[r] \arrow[dr] & R_g \arrow[d, phantom, "\shortparallel"] \\ & (R_f)_g = R_{fg} \end{tikzcd}$$

If X is a top space, $\mathcal{B}$ a basis for top

$\mathcal{F} : \mathcal{B}^{op} \longrightarrow \text{Set}$ such that

Prop 2.2
1. if $\{U_i\}$ covers U, U_i, U in $\mathcal{B}$ and $f, f' \in \mathcal{F}(U)$ s.t. $f|_{U_i} = f'|_{U_i} \Rightarrow f = f'$

2. if $\{U_i\}$ covers U, $U, U_i \in \mathcal{B}$ and $f_i \in \mathcal{F}(U_i)$ s.t. $f_i|_V = f_j|_V$ for all $V \subset U_i \cap U_j$, V in $\mathcal{B}$

$\Rightarrow \exists f \in \mathcal{F}(U)$ s.t. $f|_{U_i} = f_i$

Then $\exists! \tilde{\mathcal{F}}$ sheaf on X s.t. $\tilde{\mathcal{F}}(U) = \mathcal{F}(U)$ for U in $\mathcal{B}$

and
$$\begin{tikzcd} \tilde{\mathcal{F}}(U) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{\mathcal{F}}(V) \arrow[d, equal] \\ \mathcal{F}(U) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$
U, V in $\mathcal{B}$.

Pf:

Define $\tilde{\mathcal{F}}(U)$ as follows

if $\{U_i\}$ covers U U_i in $\mathcal{B}$, then

set $\mathcal{F}(\{U_i\}) = \{ (f_i)_{f_i \in \mathcal{F}(U_i)} \mid f_i|_V = f_j|_V \text{ for all } V \text{ in } \mathcal{B} \text{ in } U_i \cap U_j \}$

if $\{U_i\}$ $\{V_i\}$ covers of U, $\underset{\mathcal{F}(\{U_i\})}{(f_i)} \sim \underset{\mathcal{F}(V_i)}{(g_j)}$?

if for all i, j $V \subset U_i \cap V_j$ $f_i|_V = g_j|_V$

$$\mathcal{F}(U) = \frac{\bigsqcup \mathcal{F}(\{U_i\})}{\sim}$$

given a "$\mathcal{B}$-sheaf" $\mathcal{F}$

$\exists$ $\tilde{\mathcal{F}}$ sheaf together w/ an isom

$$\mathcal{F} \xrightarrow{\sim} \tilde{\mathcal{F}}|_{\mathcal{B}}$$

s.t. given any $\mathcal{G}$ sheaf and

$\mathcal{F} \longrightarrow \mathcal{G}|_{\mathcal{B}}$ then $\exists!$ $\tilde{\mathcal{F}} \to \mathcal{G}$

s.t.
$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \tilde{\mathcal{F}}|_{\mathcal{B}} \arrow[d] \\ & \mathcal{G}|_{\mathcal{B}} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Funct}(\mathrm{Op}(X)^{op}, \mathrm{Sets}) \arrow[d] & \mathrm{PreShf} \arrow[l, hook'] & \mathrm{Shf} \arrow[l, hook'] \arrow[dl] \\ \mathrm{Funct}(\mathcal{B}^{op}, \mathrm{Sets}) & \mathcal{B}\mathrm{Shv} \arrow[l, hook'] & \end{tikzcd}$$

$\mathcal{F} \mapsto \mathcal{G}$