Localization Reminder

R comm. ring $\quad S \subset R$ a multiplicative set (nonempty)
(i.e. S is closed under multiplication)

Def $R_S = R[S^{-1}] = \left\{ \frac{r}{s} \mid r \in R,\ s \in S \right\} / \sim$

$$\frac{r}{s} \sim \frac{r'}{s'} \iff \exists s'' \in S \text{ s.t. } s''(s'r - r's) = 0$$

$$\frac{r}{s} + \frac{r'}{s'} \equiv \frac{rs' + r's}{ss'}$$

$$\boxed{R_f = R[\{1, f, f^2, f^3, \ldots\}]}$$

$$\varphi : R \longrightarrow R_S \qquad s \in S$$
$$r \longmapsto \frac{rs}{s}$$

kernel of $\varphi = \{ r \in R \mid s'r = 0 \text{ some } s' \in S \}$

if $P \triangleleft R$ prime, and $S \cap P = \emptyset$
then $\underbrace{\varphi(P) R_S}_{(PR_S)}$ prime in R_S

and if $Q \triangleleft R_S$ prime, then $\underbrace{\varphi^{-1} Q}_{(Q \cap R)}$ prime in R

Induces a bijection between the primes of R_S and primes in R disjoint from S.

Previously, defined $\operatorname{Spec} R = \{\text{primes in } R\}$

if $f \in R$, regarded $\operatorname{Spec} R_f \subset \operatorname{Spec} R$

consisting of primes not containing f.

Cor: if $f \in R$ not nilpotent then

$\exists$ prime $P \subset R$ not containing f.

Pf: R_f not the 0 ring. $\frac{1}{1} \sim \frac{0}{1} \Leftrightarrow f^l \cdot 1 = 0$

$\Rightarrow R_f$ has a maximal ideal $\Rightarrow$ prime $\Rightarrow$ prime in R not containing f

---

Last time, to define the sheaf $\mathcal{O}_{\operatorname{Spec} R}$,

we defined it on a basis $\mathcal{O}_{\operatorname{Spec} R}(D_f) = R_f$

$D_f = \{P \in \operatorname{Spec} R \mid f \notin P\}$

Want: if $r \in \mathcal{O}_{\operatorname{Spec} R}(D_f)$ and D_f covered by $\bigcup D_{f_i}$

then $r|_{D_{f_i}} = 0$ all $i \Rightarrow r = 0$.

What does it mean to say $D_f = \bigcup D_{f_i}$

$D_f \supset D_g \quad \forall P$ prime

$P \not\ni g \Rightarrow P \not\ni f$

$\forall P$ prime

$g \in P \Leftarrow f \in P$

$\langle g \rangle \subset P \Leftarrow \langle f \rangle \subset P$

$$g \in \bigcap_{P \supset \langle f \rangle} P = \sqrt{\langle f \rangle}$$

$D_f \supset D_g \Leftrightarrow g \in \sqrt{\langle f \rangle}$

$g^l = af$

$R_f \to R_g$

$$\frac{r}{f^m} \longmapsto \frac{r\,a^m}{(af)^m} = \frac{ra^m}{g^{lm}}$$

$$\boxed{\begin{array}{l}\text{Reminder} \\ \sqrt{I} = \bigcap_{P \supset I} P \\ = \{r \mid r^i \in I \text{ some } i\}\end{array}}$$

$\bigcup D_{f_i} = D_f$

$P \in D_f \iff P \in D_{f_i}$ some i

$f \notin P \iff f_i \notin P$ some i

$f \in P \iff f_i \in P$ all i

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

$\langle f \rangle \subset P$ $\qquad\qquad$ $\langle f_i \rangle \subset P$

$\sqrt{\langle f \rangle} = \sqrt{\langle f_i \rangle}$

$\bigcup D_{f_i} \subset D_f$

$f_i \in \sqrt{\langle f \rangle}$ all i

$\left(D_f \subset \bigcup D_{f_i} \iff f \in \sqrt{\langle f_i \rangle} \right)$

$f^m = \sum g_i f_i$

Given $r \in R_f$ suppose $r \mapsto 0$ in R_{f_i} each i

$\sqrt{\langle f \rangle} = \sqrt{\langle f_i \rangle}$

want to show $r = 0$.

$f_i^{l_i} = a_i f$ some $a_i \in R$

$r = \frac{b}{f^m}$, $b \in R$

$r \mapsto \frac{b a_i^m}{(a_i f)^m} = \frac{b a_i^m}{f_i^{l m}}$

if this is 0 in R_{f_i} then $\Rightarrow$

$$\frac{b a_i^m}{(a_i f)^m} = \frac{0}{(a_i f)^m}$$

$$\Rightarrow f_i^{N_i} \underbrace{(a_i f)^m}_{a_i f = f_i^{l_i}} b a_i^m = 0$$

mult. by f^m

$$f_i^{N_i} \underbrace{(a_i f)^m}_{f_i^{l_i}} b \underbrace{(a_i^m f^m)}_{f_i^{l_i m}} = 0$$

$$f_i^{M_i} b = 0. \quad \text{all } i$$

$$f_i^{M_i} \in \operatorname{ann}_R b \quad \text{all } i$$

$$\langle f_i^{M_i} \rangle \subseteq \operatorname{ann}_R b$$

$$\sqrt{\langle f_i^{M_i} \rangle} \subseteq \sqrt{\operatorname{ann}_R b}$$

$$\shortparallel$$

$$\sqrt{\langle f_i \rangle}$$

$$\shortparallel$$

$$f \in \sqrt{\langle f_i \rangle} \Longrightarrow f^N b = 0$$

$$\Rightarrow \frac{b}{1} = 0 \text{ in } R_f$$

$$\Rightarrow \frac{b}{f^m} = 0 \text{ in } R_f$$

$$\shortparallel$$

$$r \quad ∎.$$

More about Spec R

We now "have" that $\mathcal{O}_{\operatorname{Spec} R}$ is a sheaf of rings on Spec R. $\Rightarrow (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a ringed space.

Notation: if $P \triangleleft R$ prime
$R_P = R_{R \setminus P}$

Lemma $\mathcal{O}_{\operatorname{Spec} R, P} = R_P$

Pf: $\mathcal{O}_{\operatorname{Spec} R, P} = \varinjlim_{U \ni P} \mathcal{O}_{\operatorname{Spec} R}(U)$

but the opens D_f containing P are cofinal, so

$$\mathcal{O}_{\operatorname{Spec} R, P} = \varinjlim_{D_f \ni P} \mathcal{O}_{\operatorname{Spec} R}(D_f) = \varinjlim_{f \notin P} R_f = R_P$$

exercise.

$\square$

So $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Prop (2.3) If R, S rings then

$$\operatorname{Hom}_{\text{rings}}(R, S) = \operatorname{Hom}_{\substack{\text{loc. ringed}\\ \text{spaces}}}(\operatorname{Spec} S, \operatorname{Spec} R)$$

not the proof: if $\varphi : R \longrightarrow S$

induces $\varphi^{-1} : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$

$P \longmapsto \varphi^{-1} P$

suppose $\psi : \operatorname{Spec} S \to \operatorname{Spec} R$

corresponding to it

then $\psi = \varphi^{-1}$ on top spaces

$\psi^{\#} : \mathcal{O}_{\operatorname{Spec} R} \to \psi_* \mathcal{O}_{\operatorname{Spec} S}$

on global sections

$$\begin{array}{ccc} \mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) & \to & \psi_* \mathcal{O}_{\operatorname{Spec} S}(\operatorname{Spec} R) \\ \| & & \| \\ R & & \mathcal{O}_{\operatorname{Spec} S}(\operatorname{Spec} S) \\ & & \| \\ & & S \end{array}$$

will be φ

Def An affine scheme is a locally ringed space isomorphic to $\operatorname{Spec} R$ some R

Def A scheme is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{U_i\}$ of X

such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme.

Ex: $\mathbb{A}^1_k = \operatorname{Spec} k[x]$

$\mathbb{A}^n_R = \operatorname{Spec} R[x_1, \dots, x_n]$

$\mathbb{A}^1_k$ ← points are prime ideals in $k[x]$ ← PID

pts of $\mathbb{A}^1_k$ $\updownarrow$ $0 \cup \{\text{irred monic polys}\}$

0, $(f(x))$ $f(x)$ monic irred poly.

if $k = \bar{k}$ irred $\leftrightarrow$ $(x-a)$ $a \in \bar{k}$

if $g(x) \in k[x]$ "regular func on $\mathbb{A}^1_k$"

$$\mathcal{O}_{\mathbb{A}^1_k}(\mathbb{A}^1_k) \longrightarrow \mathcal{O}_{\mathbb{A}^1_k,(x-a)}$$

$$\mathcal{O}_{\mathbb{A}^1_k}(\mathbb{A}^1_k) = k[x]$$

$$k[x]_{(x-a)} \to \mathcal{O}_{\mathbb{A}^1_k,(x-a)} / \mathfrak{m}_{(x-a)}$$

$$g \in k[x] \longrightarrow k[x]/(x-a), \quad g \mapsto g(a)$$

$$k[x]_{(x-a)} \longrightarrow k[x]_{(x-a)} / (x-a)k[x]_{(x-a)}$$

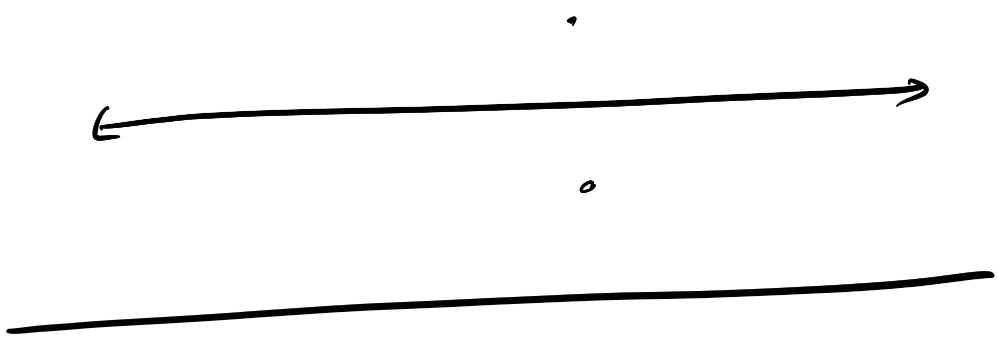
$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R_p \arrow[r] & R_p/pR_p = k \\ & R/p \arrow[ur] & \end{tikzcd}$$

$$R_p/pR_p = \operatorname{frac}(R/p)$$

$\mathbb{A}^1_{\mathbb{R}} \ni (x^2+2)$

$$\mathbb{R}[x] \longrightarrow \frac{\mathbb{R}[x]}{x^2+2} = \mathbb{R}(\sqrt{-2}) = \mathbb{C}$$

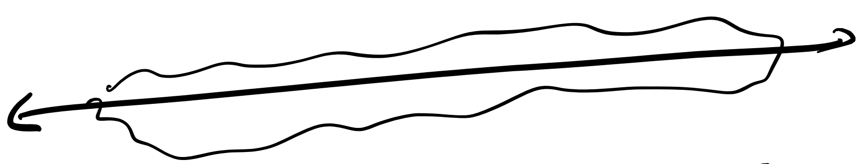
$$g(x) \longmapsto g(\sqrt{-2})$$



$\mathbb{A}^1_k \ni 0$

$$k[x] \xrightarrow{\text{evaluation}} k(x)$$

$$g(x) \longmapsto g(x)$$



$$\mathbb{Q}(x) \subset \overline{\mathbb{Q}(x)} \simeq \mathbb{C}$$