R ring   Spec R q.compact top space.

(showed if D_f covered by D_{g_i} then it is covered by a finite subcollection of D_{g_i}'s.)

$f = 1$ $D_f = \operatorname{Spec} R$ U_i cover D_{g_j} basis

can refine U_i cover via D_{g_j}'s....

$R = \prod_{i=1}^{\infty} k$ k field. $(a_i)_{i \in \mathbb{Z}_{>0}}$ $a_i \in k$.

$f_i = (a_j)$ $a_j = \delta_{ij}$

$R_{f_i} = k \leftarrow$ ith coord.

$\ker R \longrightarrow R_{f_i}$ is (a_j) s.t. $a_i = 0$.

Spec R 

Looks like Spec R_{f_i} cover Spec R all open. but no finite subcollection covers.. ?!?

Recall:

Def A scheme is a locally ringed space $(X, \mathcal{O}_X)$ s.t. $\exists$ open cover $\{U_i\}$ of X with

$$(U_i, \mathcal{O}_X|_{U_i}) \underset{\substack{\text{loc.}\\ \text{ringed spaces}}}{\simeq} (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$$ for some rings A_i.

Construction by gluing

If $(X_i, \mathcal{O}_{X_i})$ $i = 1, 2$ are schemes, and

if $U_i \subset X_i$ open subsets, and

$$\varphi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \xrightarrow{\sim} (U_2, \mathcal{O}_{X_2}|_{U_2})$$

then $\exists!$ a scheme $(X, \mathcal{O}_X)$ together with morphisms

$$\begin{tikzcd}
(U_1, \mathcal{O}_{X_1}|_{U_1}) \arrow[r] \arrow[d, "\varphi"] & (X_1, \mathcal{O}_{X_1}) \arrow[dr] & \\
(U_2, \mathcal{O}_{X_2}|_{U_2}) \arrow[r] & (X_2, \mathcal{O}_{X_2}) \arrow[r] & (X, \mathcal{O}_X)
\end{tikzcd}$$

where $X = X_1 \sqcup_\varphi X_2$

$$\begin{tikzcd}
 & U_1 \arrow[r] \arrow[dl, "\varphi"'] \arrow[d] & X_1 \arrow[d] \\
U_2 \arrow[r] & X_2 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & X
\end{tikzcd}$$

$$X = X_1 \sqcup X_2 /\sim$$

$x_1 \sim \varphi(x_1)$ if $x_1 \in U_1$

Shortcut proof of $\exists, !$,
sheaves are determined by values on a basis
use prior formulation of sheaves on basis. ($\mathcal{B}$-sheaf)

$$\mathcal{B} = \{\text{opens s.t. } U \subset X_1 \text{ or } X_2\}$$

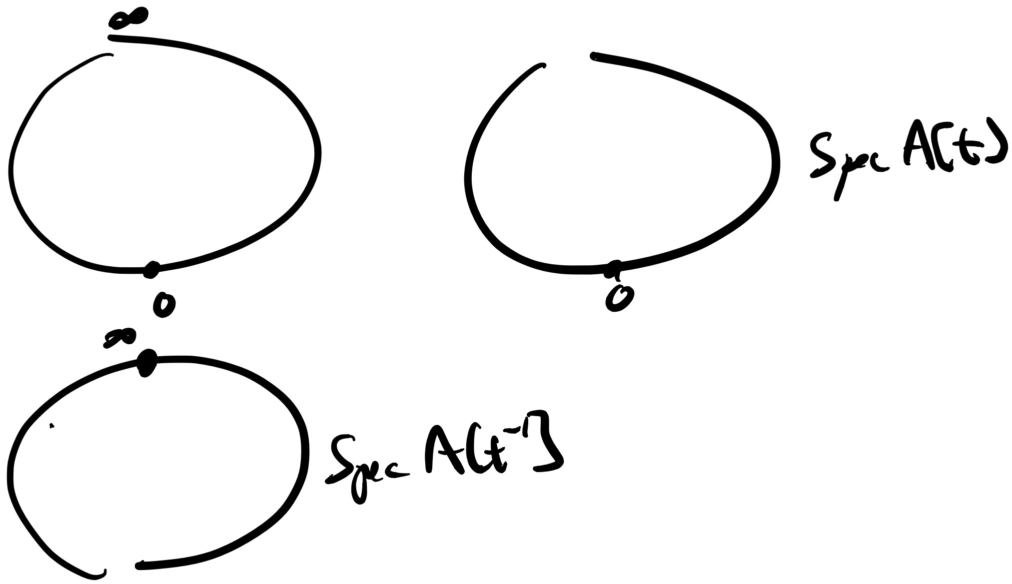
<u>Example</u> $\mathbb{P}^1_A$ A comm. ring. x $(x-a)$

$$X_1 = \operatorname{Spec} A[x] \supset U_1 = \operatorname{Spec} A[x, x^{-1}]$$

$$X_2 = \operatorname{Spec} A[y] \supset U_2 = \operatorname{Spec} A[y, y^{-1}]$$

$$A[x, x^{-1}] \longrightarrow A[y, y^{-1}]$$
$$x \longmapsto y^{-1}$$
$$x^{-1} \longmapsto y$$

$$\operatorname{Spec} A[t] > \operatorname{Spec} A(t, t^{-1})$$
$$\downarrow \text{"id"}$$
$$\operatorname{Spec} A[t^{-1}] > \operatorname{Spec} A[t, t^{-1}]$$



$$\boxed{\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) = k}$$

HW.

$\operatorname{Spec} k = \mathbb{P}^1_k$

$\operatorname{Spec} k[x]$

Proj

Suppose $S_\bullet = \bigoplus_{i=0}^{\infty} S_i \qquad S_i S_j \subset S_{i+j}$

Language: $s \in S_\bullet$ is homogeneous if $s \in S_i$ some i.
i = degree.

an ideal $I \trianglelefteq S_\bullet$ is homogeneous iff
I is gen. by hom. elements iff
$I = \bigoplus_{i=0}^{\infty} I_i$ where $I_i = I \cap S_i$

<u>Useful facts</u>: if $S_\bullet$ is a graded ring,

- $f \in S_\bullet$ is homogeneous, then $S_\bullet[f^{-1}]$ is graded. (graded by $\mathbb{Z}$)

- if $I \triangleleft S_\bullet$ is hom. then $S_\bullet/I$ is graded.

<u>Def</u> $\operatorname{Proj} S_\bullet$ is a scheme (loc. ringed space)

the top space: set of hom. prime ideals

($P \triangleleft S_\bullet$ prime, hom)

if $I \triangleleft S_\bullet$ hom, $Z(I) = \{P \in \operatorname{Proj} S_\bullet \mid I \subset P\}$

these define closed sets of "Zariski top".

if $f \in S_i$ homogeneous deg i,

$$D^+(f) = (\operatorname{Proj} S_\bullet) \setminus Z(f)$$

$$Z(f) = Z(fS_\bullet)$$

$$D^+(f) = \{P \in \operatorname{Proj} S_\bullet \mid f \notin P\}$$

$$\mathcal{O}_{\operatorname{Proj} S_\bullet}(D^+(f)) = S_{(f)}$$

Def $S_{(f)} = \left\{ \frac{g}{f^i} \in S_0[f^{-1}] \;\middle|\; \deg \frac{g}{f^i} = 0 \right\}$

Recall: $\mathbb{P}(V)$ if $x_1, \dots, x_n$ coord fcns on V
w/r/t to some basis

$k[x_1, \dots, x_n]$ polys on $V = \mathbb{A}(V)$

$\mathbb{P}(V) =$ lines in V $\quad \ell = \langle (a_1, \dots, a_n) \rangle$

given $\quad g \in k[x_1, \dots, x_n]$

$$\cancel{g(\ell) = g(a_1, \dots, a_n)}$$

$g \in k[x_1, \dots, x_n][f^{-1}] \quad f \deg i$

$$g = \frac{h(x)}{f(x)} \quad h(x) \deg i$$

$$g(\ell) = g(\lambda a_1, \dots, \lambda a_n) = \frac{\lambda^i h(\vec{a})}{\lambda^i f(\vec{a})}$$

$$\mathcal{O}_{\mathrm{Proj}\, S_0}\big|_{D^+(f)} = \mathcal{O}_{\mathrm{Spec}\, S_{(f)}}$$

$$\left(D^+(f), \mathcal{O}_{\mathrm{Proj}\, S_0}\big|_{D^+(f)}\right)$$

$$\cong$$

...

Spec $S_{(f)}$

Altrnate description

$$\operatorname{Proj} S_\bullet = \bigcup \operatorname{Spec} S_{(f)}$$

via gluing.

Next set of topics

Properties of schemes & morphisms between them.

Def if S is a scheme, an S-scheme is a scheme X together with a morphism $X \to S$ ("the structure morphism")

geometrically:

$$\begin{array}{c} X \\ \downarrow{\scriptstyle \pi} \\ S \end{array}$$

"spaces"

we think of X as a varying family of spaces, parametrized by S.

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ \downarrow & & \downarrow \\ s & \xrightarrow[\text{pt}]{} & S \end{array}$$

algebraically: $S = \operatorname{Spec} A$

to say that X is an A-scheme

means that all rings $\mathcal{O}_X(U)$ are A-algebras and restriction maps are A-algebra maps.

$$\begin{tikzcd} U \arrow[r] \arrow[dr, "f"'] & X \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd}$$

$$\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \xrightarrow{f^\#} f_* \mathcal{O}_U(\operatorname{Spec} A)$$

$$\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A, \quad f_*\mathcal{O}_U(\operatorname{Spec} A) = \mathcal{O}_U(U) = \mathcal{O}_X(U)$$

---

morphism of S-schemes

$$\begin{matrix} X & & Y \\ \downarrow & & \downarrow \\ S & & S \end{matrix}$$

are morphisms $X \to Y$ s.t.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

commutes.

Rem (Prop 2.6)

there's a full faithful embedding of varieties over an alg closed field k into the cat. of schemes over Spec k.

"quasi-projective varieties"

<u>Next step:</u>

properties of schemes inherited from rings.

- if a ring has prop P which is preserved under localization then can say a scheme has prop P if all affines in (some) (all) covers have prop...

<u>Ex:</u> A scheme is reduced if ← no nilpotents

- each local ring is reduced (stalk of $\mathcal{O}_X$)

equiv. • $\exists$ cover of affines w/ each reduced.

equiv. • $\mathcal{O}_X(U)$ reduced all U.

Def A scheme X is integral iff each $\mathcal{O}_X(U)$ is a domain
(not the same as covered by $\operatorname{Spec} A_i$'s, each A_i domain)

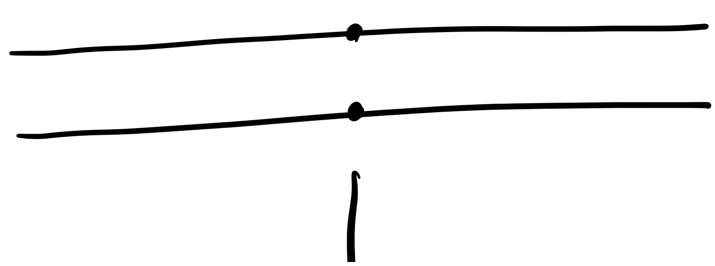
$\operatorname{Spec}(k \times k)$ covered $\operatorname{Spec} k \cup \operatorname{Spec} k$

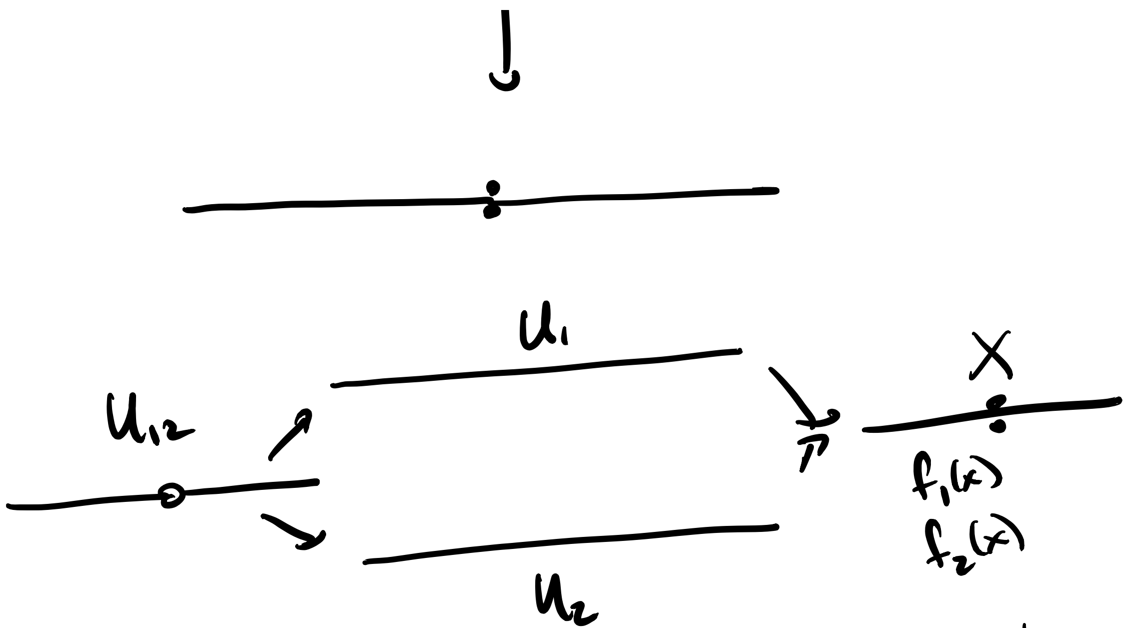
$A = k \times k$ not a domain.

$$S_\bullet / I \xleftarrow{\text{deg } 0 \text{ hom map}} S_\bullet$$

$S_\bullet = \bigoplus S_i$ $\qquad I = \bigoplus I_i$

$$S_\bullet / I = \bigoplus \underbrace{S_i / I_i}_{\text{deg } i \text{ part}}$$





$$\mathcal{O}_X(U_1), \quad \mathcal{O}_X(U_2)$$
$$a_1 \in \mathcal{O}_X(U_1), \quad a_2 \in \mathcal{O}_X(U_2)$$

$$a_1|_{U_{12}} = a_2|_{U_{12}}$$

$$\exists!\ a$$

$$\mathcal{O}_X(U_1) \longrightarrow \mathcal{O}_X(U_{12})$$
$$k[x] \hookrightarrow k[x, x^{-1}]$$