Proj Revisited

$a_0 \neq 0$ $\quad [1; \frac{a_1}{a_0}; \dots; \frac{a_n}{a_0}] = [1; b_1, \dots]$

$\mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n \longleftrightarrow \mathbb{A}^n = D^+(x_0)$

$(a_0, a_1, \dots, a_n) \longmapsto [a_0; a_1; \dots; a_n]$ $\quad [1; b_1; b_2; \dots; b_n]$

$\downarrow x_i$ $\rightsquigarrow$ "hom coords" x_i $\quad$ $(b_1, b_2, \dots, b_n)$

$x_i = 0$ makes sense $\quad$ affine coords u_i

a_i $\quad b_i$

$$u_i = \frac{x_i}{x_0}$$

Closed subsets of $\mathbb{P}^n$

$\updownarrow$

Zeros of hom polys

$Z(f_1, \dots, f_r)$ $\quad$ unless $Z(I) = \text{empty}$

$\quad$ i.e. $Z(I) \subset Z(\langle x_i \rangle)$

$Z(I)$ $\quad I = \langle f_i \rangle$ $\quad I \supset \langle x_i \rangle = S_+$

"irrelevant ideal"

points of proj S

$\updownarrow$

hom. primes which don't contain S_+

(in particular if $S_+ \subset I$ $\quad Z(I) = \{P \in \operatorname{Proj} S \mid I \subset P\} = \emptyset$)

$S_+ \subset I \subset P \Rightarrow P \notin \operatorname{Proj} S$

Properties of schemes

X is a scheme

Def X is connected (irreducible) if its underlying top. space is connected (irreducible)

Def X is reduced if $\mathcal{O}_X(U)$ is a reduced ring, all U.
(equivalently, $\mathcal{O}_{X,p}$ is reduced all p)

Def X is integral if $\mathcal{O}_X(U)$ is a domain for all U.

Prop X is integral iff X is reduced & irreducible.

Pf: X integral $\Rightarrow$ X is reduced

to see X irreducible suppose $X = X_1 \cup X_2$ closed

set $U = X \setminus (X_1 \cap X_2)$ open $\Rightarrow$ U irred.

$U_i = X \setminus X_i$ $\quad U = U_1 \sqcup U_2$ ← disj. union.

is a III of sets → $\quad \| \quad X_j \setminus (X_i \cap X_j)$

let $s_1 = 1$ on $\mathcal{O}_X(U_1)$ $\quad s_2 = 0$ on $\mathcal{O}_X(U_2)$

sections on a cover of $U_1 \cup U_2$
agree on overlap $U_1 \cap U_2 = \emptyset$

$\exists\, s \in \mathcal{O}_X(U)$ s.t. $s|_{U_1} = 1 \quad s|_{U_2} = 0$

similarly $\exists\, s' \in \mathcal{O}_X(U)$ s.t. $s'|_{U_1} = 0 \quad s'|_{U_2} = 1$

$ss' = 0$ when restricted to U_1 & U_2

but these cover $\Rightarrow ss' = 0$ (separated)

$s, s' \neq 0 \Rightarrow \mathcal{O}_X(U)$ not a domain ⚡.

Suppose X is reduced & irred. why is X integral?

Suppose we have $fg = 0$ for $f, g \in \mathcal{O}_X(U)$

$U \subset X$ open (and so irred)

Let $Z_f = \{P \in U \mid [f] \in \mathcal{O}_{X,P}$ is in $m_P\}$

is closed in U. (HW)

similarly Z_g.

given P, we know $[f][g] \in \mathcal{O}_{X,P}$

$$[f][g] = [fg] = 0$$

is in $m_P \Rightarrow [f]$ or $[g] \in m_P$

$Z_f \cup Z_g = U$. U irred $\Rightarrow Z_f = U$ or $Z_g = U$.

WLOG $Z_f = U$.

Let $\operatorname{Spec} R \subset U$ be an affine open subscheme.

let $\tilde{f} = \operatorname{res}_{U, \operatorname{Spec} R} f$ ETS $\tilde{f} = 0$ (all R's)

$\Rightarrow$ (sep sheaf) $f = 0$.

for $p \in \operatorname{Spec} R \subset X$

$$\begin{tikzcd}
\tilde{f} \in R \arrow[r] \arrow[dr] & R_p \arrow[r] & R_p/pR_p \\
& R/p \arrow[ur, hook] &
\end{tikzcd}$$

$R_p = \mathcal{O}_{X,p}$, $R_p/pR_p = m_p$

exercise

R/p ← domain

R_p/pR_p ← field

frac field

i.e.

$$\boxed{R_p/pR_p = \operatorname{frac}(R/p)}$$

$\overset{\text{diagram}}{\Longrightarrow}$ $\tilde{f} \in P$ all $P \in \operatorname{Spec} R$

$\Rightarrow \tilde{f} \in \sqrt{R} \Rightarrow \tilde{f}^n = 0$ in $R = \underbrace{\mathcal{O}_X(\operatorname{Spec} R)}_{\text{reduced}}$

$\Rightarrow \tilde{f} = 0$ Δ.

<u>Def</u> X is locally Noetherian if $\exists$ cover by affine open sets $U_i = \operatorname{Spec} R_i$ s.t. each R_i is a Noeth. ring.

<u>ex</u>: $X = \coprod^{\infty} \operatorname{Spec} k$

<u>Lemma</u> $\operatorname{Spec} R$ is loc. Noeth iff R is Noeth.

$\Leftarrow$ ✓

$\Rightarrow$?

Sublemma If $f_1, \ldots, f_r \in R$ generate the unit ideal and R_{f_i} is Noeth each $i \Rightarrow R$ Noeth.

Subsublemma If $I \triangleleft R$ and f_i as above then $I = \bigcap_i \left(R \cap (IR_{f_i}) \right)$ $\qquad R \to R_{f_i}$

Pf: $\subset$ ✓

$\supset$: if $x \in R$ and $\frac{x}{1} \in IR_{f_i}$ all i

then for some $x_i \in I$ $\frac{x}{1} = \frac{x_i}{f_i^{n_i}}$

$\Leftrightarrow \left(x f_i^{n_i} - x_i\right) f^{m_i} = 0$

i.e. $x f_i^{n_i + m_i} = x_i f^{m_i} \in I$

so $x f_i^{n_i + m_i} \in I$ set $N = \max\{n_i + m_i\}$

then $x f_i^N \in I$ all i

notice: $\langle f_i \rangle = R \Leftrightarrow \langle f_i^N \rangle = R$

$\Rightarrow 1 \in \langle f_i^N \rangle$ $\qquad 1 = \sum c_i f_i^N$

$x = x \sum c_i f_i^N$

$= \sum c_i \underbrace{x f_i^N}_{I} \in I$

$\Rightarrow x \in I$ $\supset$ ✓ $\square$

<u>Sublemma</u> If $f_1, \dots, f_r \in R$ generate the unit ideal and R_{f_i} is Noeth each $i \Rightarrow R$ Noeth.

<u>Pf:</u> given $I_1 \subset I_2 \subset \dots$ asc. chain.

get asc. chains:

$$\left(R \cap (I_1 R_{f_i})\right) \subset \left(R \cap (I_2 R_{f_i})\right) \subset \dots$$

know $I_1 R_{f_i} \subset I_2 R_{f_i} \subset \dots$ stabilizes (R_{f_i} Noeth)

$\Rightarrow$ stabilizes.

so $\exists$ M s.t. all chains for each $i = 1, \dots, n$ stabilizes after M

$$\Rightarrow \underbrace{\bigcap_i \left(R \cap (I_1 R_{f_i})\right)}_{= I_1} \subset \underbrace{\bigcap_i \left(R \cap (I_2 R_{f_i})\right)}_{= I_2} \subset \dots$$

stabilizes after M so I_j's stabilize.

$\square$.

__Pf of Lemma:__

Suppose $\operatorname{Spec} R_i$ cover $\operatorname{Spec} R$ w/ R_i Noeth.

WTS R Noeth. $\quad \operatorname{Spec} R_i = \bigcup_j D_{f_{ij}}$

if we choose $f \in R$ w/ $D_f \subset \operatorname{Spec} R_i$

then if $\bar{f} = \operatorname{res}_{\operatorname{Spec} R, \operatorname{Spec} R_i} f$, $D_f = D_{\bar{f}}$

$$D_f = \{P \in \operatorname{Spec} R \mid f \notin P\} = \{P \in \operatorname{Spec} R_i \subset \operatorname{Spec} R \mid \ldots\}$$

$$= \{P \in \operatorname{Spec} R \mid [f] \in \mathcal{O}_{\operatorname{Spec} R, P},\ [f] \notin m_P\}$$

$$= \{P \in \operatorname{Spec} R_i \mid [\bar{f}] \in \mathcal{O}_{\operatorname{Spec} R_i, P},\ [\bar{f}] \notin m_P\}$$

$$= D_{\bar{f}}$$

__Note:__ if B is Noeth then B_f is also Noeth.

$$\left(B_f = \frac{B[x]}{xf - 1}\right)$$

So $D_{\bar{f}} = \operatorname{Spec} (R_i)_{\bar{f}}$

and $(R_i)_{\bar{f}}$ is Noeth since R_i is.

and choosing f's s.t. D_f's refine cover $\operatorname{Spec} R_i$

then we get a cover by affines of the form

$D_{f_j} = \operatorname{Spec} R_{f_j}$

R_{f_j}'s Noeth.

$ f_j$
 ||
$D_{f_j} \subset \operatorname{Spec} R_i$

Recall: cov'g means $\langle f_j \rangle = R$, $\Leftrightarrow$ $1 = \sum c_j f_j$

$\Rightarrow$ can restrict to get a finite subcover.

Now, done by sublemma.

<u>Def</u> X is Noetherian if X is locally Noetherian and quasi-compact.

ex. $\coprod^{\infty} \operatorname{Spec} k$

<u>Def</u> An A-algebra B is of finite type if it is finitely generated - i.e. $B = A[x_1, \dots, x_n] / I$

(note I need not be f.g. if A non Noeth)

<u>Def</u> An A-algebra B is finite if it is finitely generated as an A-module

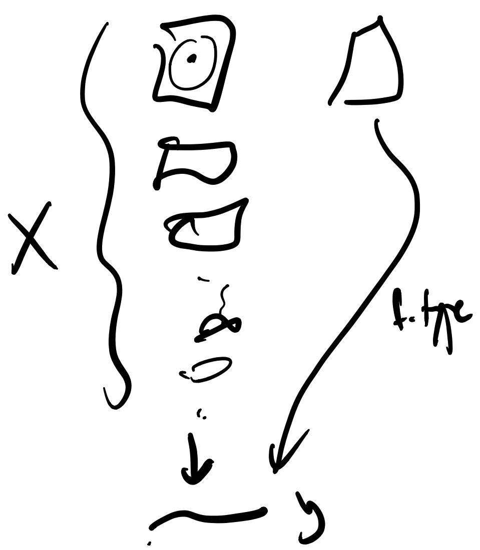
<u>example:</u> $B = A[x]/f$ ← monic "integral ext."

Def $f: X \longrightarrow Y$ (think "$\mathcal{O}_X$ an $\mathcal{O}_Y$-alg")

is locally finite type if we can cover Y by affines $U_i = \operatorname{Spec} A_i$ such that $f^{-1}(U_i)$ can be covered by affines $V_{ij} = \operatorname{Spec} B_{ij}$ s.t. B_{ij} is finite type over A_i

We say f is finite type if $f_i^{-1}(U_i)$ can be covered by a finite # of such V_{ij} as above.

(translation f.type = relatively finite dimensional)



Def $f: X \longrightarrow Y$ is finite if $\exists$ cover $U_i = \operatorname{Spec} A_i$ of Y

s.t. $f^{-1}(U_i) = \operatorname{Spec} B_i$

B_i is a finite A_i algebra.

$\rightsquigarrow$ fibers of f are "finite sets of pts"