Fibered products

Given schemes X, Y, S, morphisms

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & S \end{array}.$$

want a scheme $X \times_S Y$, which has maps making the diagram

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

commutative

such that for

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

comm. $\exists!\ Z \to X \times_S Y$

s.t.

$$\begin{array}{cccc} Z & & & \\ & \searrow & & \\ & X \times_S Y & \longrightarrow & X \\ & \downarrow & & \downarrow \\ & Y & \longrightarrow & S \end{array}$$

commutes.

Prop $X \times_S Y$ exists.

Note: if $\exists$ such a scheme w/ maps to X, Y as above, it is unique up to canonical isom.

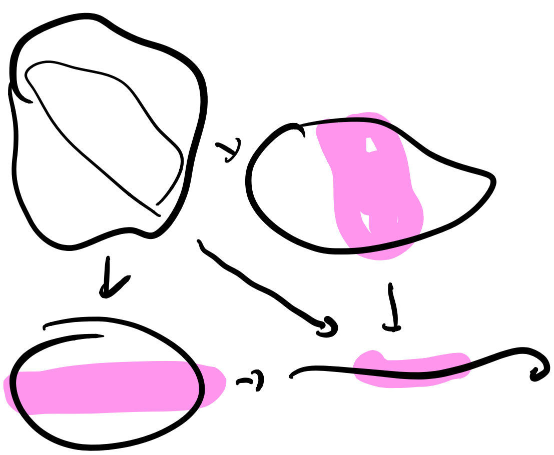
$$\left(\begin{array}{ccc} X_1 & & X_2 \\ \text{open} \uparrow & & \uparrow \text{open} \\ U & \xrightarrow{\sim} & U \end{array} \qquad \begin{array}{ccc} U & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X = X_1 \sqcup_U X_2 \end{array} \right)$$

Pf: **Remarks:** if $S' \subset S$ open

$$\begin{array}{ccc} & & X \\ & & \downarrow f \\ Y & \xrightarrow{g} & S \end{array} \qquad \begin{array}{ccc} & & f^{-1}(S') \\ & & \downarrow \\ g^{-1}(S') & \longrightarrow & S' \end{array}$$

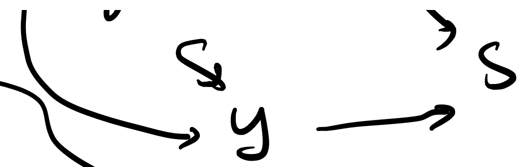
and if we can form $X \times_S Y$ then we have a map

$$X \times_S Y \xrightarrow{f \times g} S \quad \text{and} \quad (f \times g)^{-1}(S') = f^{-1}(S') \times_{S'} g^{-1}(S')$$



Pf: given

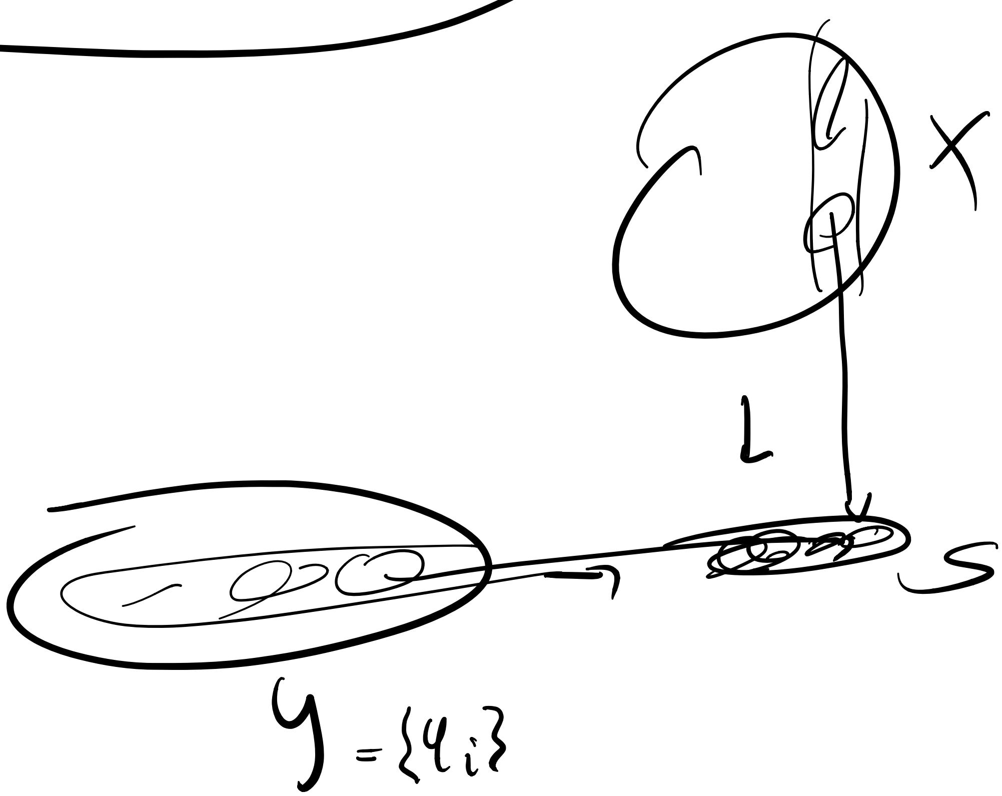
$$\begin{array}{ccccc} Z & \longrightarrow & f^{-1}(S') & \hookrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ g^{-1}(S') & \longrightarrow & S' & \hookrightarrow & \end{array}$$



Ex: $Z \xrightarrow{f} S$

f defined by g.

$Z \xrightarrow{g} S' \to S$ open inc



Affine schemes

$$X = \operatorname{Spec} A \quad Y = \operatorname{Spec} B$$

$$S = \operatorname{Spec} R$$

$$\begin{array}{ccc} Z & & \\ & \searrow^{\exists!} & \\ \downarrow & \operatorname{Spec} A \otimes_R B & \to \operatorname{Spec} A \\ & \swarrow & \downarrow \\ \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} R \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccc} \Gamma(Z, \mathcal{O}_Z) & \overset{\exists!}{\longleftarrow} A \otimes_R B & \leftarrow A \\ \uparrow & \nearrow & \uparrow \\ B & \longleftarrow & R \end{array}$$

$$\operatorname{Hom}_{Sch}(Z, \operatorname{Spec} C) = \operatorname{Hom}_{rings}(C, \Gamma(Z, \mathcal{O}_Z))$$

Ex 2.4

other idea:

if $\{X_i\}$ open cover of X &

$X_i \times_S Y$ exist

then $X \times_S Y$ exists.

... isom.

Pf: $$(X_i \cap X_j) \times_S Y \overset{\text{can id}}{=} \pi_{1,i}^{-1}(X_i \cap X_j)$$

↗ identity

opens in both

$X_i \times_S Y$

& $X_j \times_S Y$

$\pi_{1,i} : X_i \times Y \to X_i$

$\pi_{1,j} : X_j \times Y \to X_j$

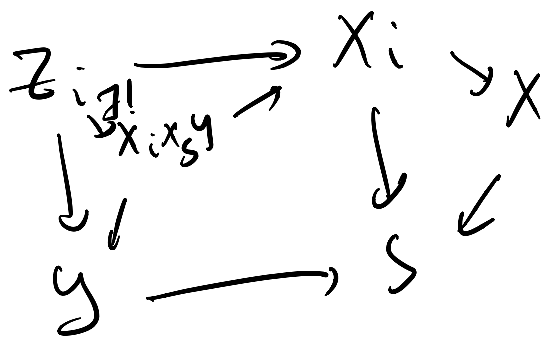
can glue to get a scheme which we call $X \times_S Y$, and this will be the fiber prod because:

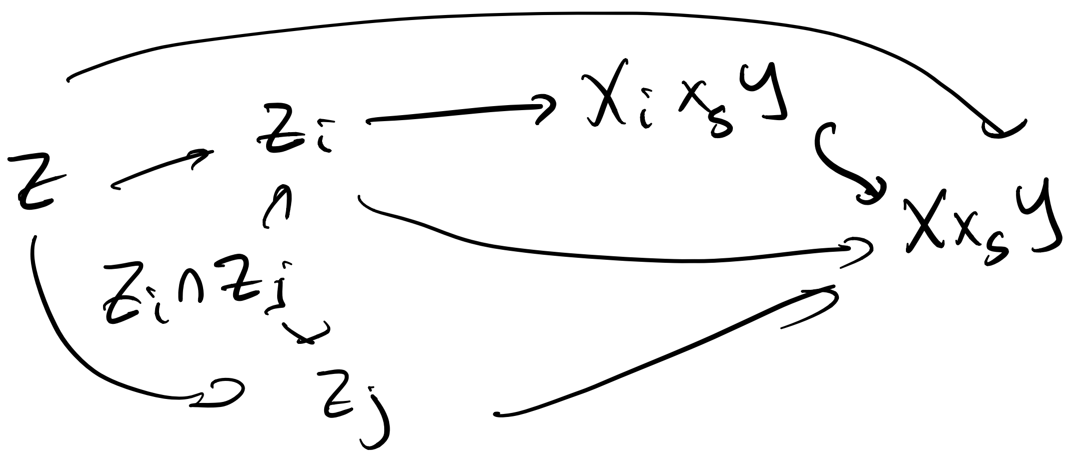
$$" \bigcup X_i \times_S Y "$$

suppose
$$\begin{array}{ccc} Z & \xrightarrow{\pi_1} & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

$Z = \bigcup Z_i \quad Z_i = \pi_i^{-1}(X_i)$

each Z_i sits in a diagram





$X \times_S Y$ exists if $X_i \times_S Y$ exists

X_i affine cover

$X_i \times_S Y$ exists if $X_i \times_S Y_j$ exists

Y_j affine cov.

first, take an affine cover of S

then make each X_i in inverse image

Y_i of given cover S

$$X_i \times_S Y_j = X_i \times_{S_e} Y_j$$

if X_i & Y_j map into S_e

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \\ \downarrow & & \downarrow g \\ Y & \longrightarrow & S \end{array}$$

$p \in X \times_S Y$, $\mathcal{O}_{X \times_S Y, p}$

$p \rightsquigarrow g^{-1}(U)$

$A \otimes_R B \ni p$

$q \in U$

Language for fiber products

Natural objects are "relative schemes"

S-scheme $X \xrightarrow{f} S$

given an S-scheme, and a map $S' \xrightarrow{\pi} S$

we can form $X \times_S S'$ which is an S' scheme

called "the base change of X to S'"

$$\begin{array}{ccc} X \times_S S' & \to & X \\ \downarrow & & \downarrow \\ S' & \to & S \end{array}$$

Natural example X, S affine
$X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$

$R \to A$
A an R-alg.

$$S' \to S \quad \longleftrightarrow \quad R \to R'$$

$$X \times_S S' = \operatorname{Spec} A \otimes_R R'$$

Fibers of a morphism

the fiber of f over p $\longrightarrow$

$$\begin{array}{ccc} X_p & \longrightarrow & X \\ \downarrow & & \downarrow f \\ p = \operatorname{Spec} \kappa(p) & \longrightarrow & S \end{array}$$

$p \in S$ point

$\kappa(p) =$ res. field $= \mathcal{O}_{S,p}/m_p$

Ex: X a scheme over $\operatorname{Spec} \mathbb{Z}$

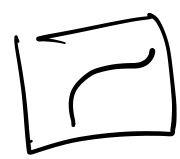
$$\begin{array}{ccc} X_{\mathbb{F}_p} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{F}_p & \longrightarrow & \operatorname{Spec} \mathbb{Z} \end{array}$$

"reductions mod p"

$$\mathbb{F}_p = \mathbb{Z}/p \longleftarrow \mathbb{Z}$$

Important point Products don't have the (fiber) product topology (at all!)

ex $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$



Separated & Proper morphisms

(analogues of Hausdorff & Complete limit pt compact)

Recall: a top space X is Hausdorff $\iff$

$X \overset{\Delta}{\hookrightarrow} X \times X$ is closed embedding

Def A morphism $X \to S$ is separated if $X \xrightarrow{\Delta} X \times_S X$ is a closed immersion.

Def A morphism $X \to S$ is proper if it is separated, finite type an universally closed.

Def: $f: X \to Y$ closed if images of closed sets are closed

Def $f: X \to Y$ univ. closed if $\forall\, Y' \to Y$ $f_{Y'}: X \times_Y Y' \to Y'$ is closed.

Recall:

Def A ring D is a valuation ring if it is a domain ($K = \text{frac}\, D$) and for every $a \in K \setminus \{0\}$ either $a \in D$ or $a^{-1} \in D$

- $K^* \xrightarrow{v} \Gamma$ ← an ordered ab. gp
  $v(ab) = v(a) + v(b)$
  - $D = \{a \in K \mid v(a) \geq 0\}$ ← id in Γ

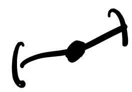
- D's are local rings w/ max'l m_D and are max'l (local) subrings of K w/ r/to domination

$$A \subset B \subset K$$

B dom A if $m_A = m_B \cap A$

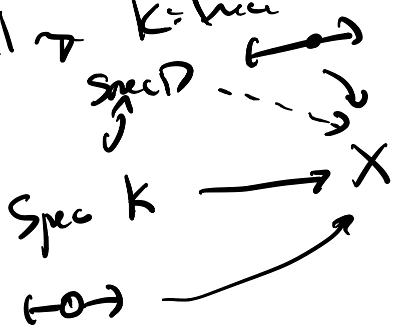
$$R \subset K$$

$m_R \dashrightarrow D$



limits in schemes

D = val r K = frac



Thm 4.3 (Valuative Criteria for Separatedness)

if $f: X \to Y$ a morphism, X Noeth

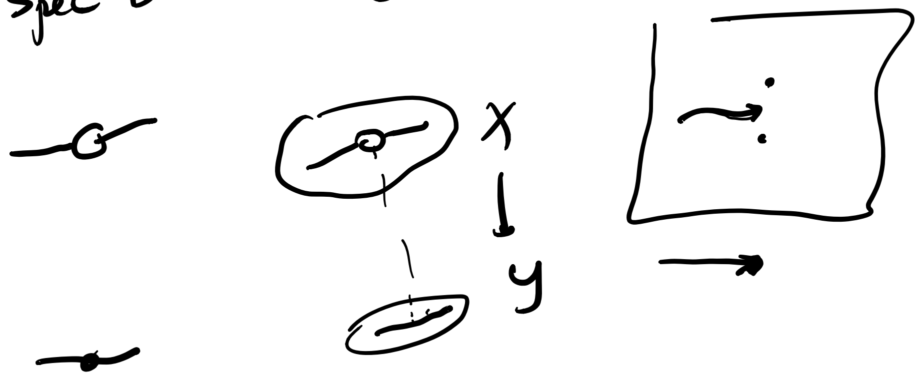
then f is sep. if and only if

for any diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} D & \longrightarrow & Y \end{array}$$

∃ at most one morphism $\operatorname{Spec} D \to X$ s.t.

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} D & \longrightarrow & Y \end{array}$$ commutes.



Thm (Valuative Criteria for properness) (4.7)

$f: X \to Y$ a morphism X noeth, f finite type

then f is proper if

for any diagram
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} D & \longrightarrow & Y \end{array}$$

∃! morphism $\operatorname{Spec} D \to X$ s.t.

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ & \nearrow & \downarrow f \end{array}$$ commutes.

$$\operatorname{Spec} D \longrightarrow Y$$

$$D \longleftarrow R$$

$$\mathfrak{p}_1 \subset \mathfrak{p}_0$$

$$\begin{array}{ccc}
K & \xleftarrow[\text{field}]{\text{frac}} & R/\mathfrak{p}_1 \\
\uparrow & \nwarrow & \cup \\
 & & \mathfrak{p}_0/\mathfrak{p}_1 \text{ prime} \\
D & \longleftarrow & (R/\mathfrak{p}_1)_{\mathfrak{p}_0/\mathfrak{p}_1}
\end{array}$$

D ↗ two pts: gen pt near $K \leftrightarrow \mathfrak{p}_1$

closed pt $\leftrightarrow \mathfrak{p}_0$

main pt of v. crit for sep. is that images of morphisms are closed $\Leftrightarrow$ closed under specialization.

$$X \longrightarrow X \times_S X$$