Separated & Proper morphisms

Def: A morphism $X \xrightarrow{f} S$ is separated if the diagonal morphism

$$\begin{tikzcd} & & X \arrow[dr] & \\ X \arrow[r, "\Delta"] \arrow[urr, bend left, "id"] \arrow[drr, bend right, "id"'] & X \times_S X \arrow[ur] \arrow[dr] & & S \\ & & X \arrow[ur] & \end{tikzcd}$$

is a closed immersion

(iff the image of Δ is closed)

Thm (4.3) (Val. crit) if $X \xrightarrow{f} S$ w/ X Noetherian then f is separated iff for any $T = \operatorname{Spec} R$ $U = \operatorname{Spec} K$

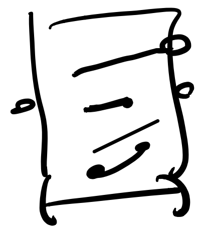
(R val ring, $K = \operatorname{frac}(R)$) and any comm. diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

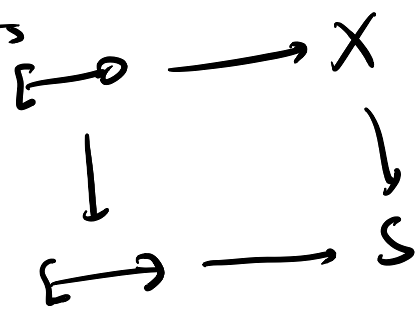
$\exists$ at most 1 $T \to X$ s.t. diag. commutes.

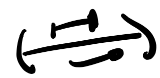
 $U = [0,1)$

 $T = [0,1]$



if Δ is closed, and we have two limits





$\alpha_1, \alpha_2 : I \longrightarrow X$ agree on $[0,1)$ then $\alpha_1 \times \alpha_2 : I \to X \times X$ takes $[0,1)$ into Δ

Δ closed $\Rightarrow$ $\alpha_1 \times \alpha_2 (1) \in \Delta$

Properness

Def $f : X \longrightarrow S$ proper if "univ. closed", separated, finite type.

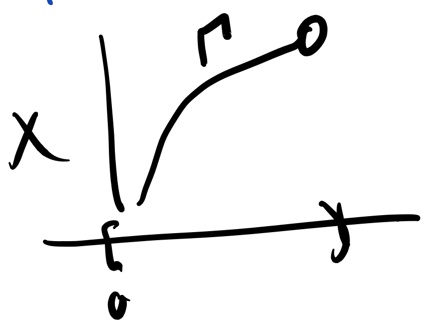
Thm (4.7) Val crit. (as sep. + uniqueness of $T \to X$)

"completeness"

WTS: given
$$\begin{tikzcd} I \arrow[r] & S \\ {[0,1)} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & X \arrow[u] \end{tikzcd}$$
then can lift uniquely to $I \to X$.

$$\begin{tikzcd} & X\times_S I \arrow[r] \arrow[d] & X \arrow[d] \\ {[0,1)} \arrow[ur, "\Gamma"] \arrow[r] \arrow[urr, bend left] & I \arrow[r] & S \end{tikzcd}$$

$x_1 \rightsquigarrow \xi_0 \mapsto \xi_1$



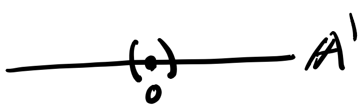
$$\begin{tikzcd} X\times I \arrow[d] \\ I \end{tikzcd}$$

consider $\overline{\Gamma} \subset X\times I$
$$\begin{tikzcd} \overline{\Gamma} \arrow[d, "\swarrow \text{closed}"] \\ I \end{tikzcd}$$

im $\overline{\Gamma}$ closed, contains $[0,1)$

val ring:

$k[x]_{(x)}$



$\operatorname{Spec} k[x]_{(x)} \setminus (x) = D_x$
$= \operatorname{Spec} k(x)$

if can take limits, why univ closed?

$$\begin{tikzcd}
{[\;]} \arrow[r] \arrow[d] \arrow[rrr, bend left] & Z \arrow[r, hook, "\text{closed}"] \arrow[d] & X \times_S T \arrow[r] \arrow[d, "f_T"] & X \arrow[d, "f"] \\
{[\;]} \ni o \arrow[r, hook] & \text{“}\mathrm{im}\, Z\text{”} \arrow[r, hook] & T \arrow[r] & S \\
m_T \arrow[u] & & {[\;]} \arrow[uur] \arrow[ur]
\end{tikzcd}$$

$$\begin{tikzcd}
{[\;]} \arrow[r] \arrow[d] & X \arrow[d] \\
{[\;]} \arrow[r] \arrow[ur, "\text{hyp.}"] & S
\end{tikzcd}$$

Will use, but not prove Lemma 4.5:

images of quasi-compact morphisms are closed if and only if they are stable under specializations.

(we say x_1 is a sp. of x_0 if $x_1 \in \overline{\{x_0\}}$)

$\lim_{t \to 0} f(t)$ $\quad$ t cont parameter

$f : S \longrightarrow X$

parameter space typically a 1 dim'l scheme w/ some closed pt $s \in S$

we define $\lim f = f(s)$

more practically speaking: start w/ $\tilde{f} : S \setminus \{s\} \to X$

we say x is a limit of $\tilde{f}$ if

$\exists f : S \longrightarrow X$ agreeing w/ $\tilde{f}$ on $S \setminus \{s\}$

s.t. $f(s) = x$.

varieties / $\mathbb{C}$

can always take $S = \operatorname{Spec} \mathbb{C}[x]_{(x)}$

or $\operatorname{Spec} \mathbb{C}[[x]]_{(x)}$ $\mathbb{C}[[x]]$

Assume val crit. for sep holds. WTS $f : X \to S$ separated.

i.e. that $\operatorname{im} \Delta$ is closed

($\Leftarrow$ $\operatorname{im} \Delta$ is closed under specialization.)

suppose we have $\xi_1 \in \operatorname{im} \Delta$ and $\xi_0 \in X \times_S X$

$= \Delta(x_1)$ $\quad \xi_0 \in \overline{\{\xi_1\}}$

want $\xi_0 = \Delta(x_0)$ some $x_0 \in X$.

Let $k(\xi_0)$ be the residue field of ξ_0

$$\begin{tikzcd} \operatorname{Spec} \xi_1 \arrow[r] \arrow[dr] & X \times_S X \\ & X \arrow[u, "\Delta"'] \end{tikzcd}$$

to say that ξ_0 is a specialization.

consider $\overline{\{\xi_1\}} \ni \xi_0$

$\mathcal{O}_{\overline{\{\xi_1\}},\xi_0}$

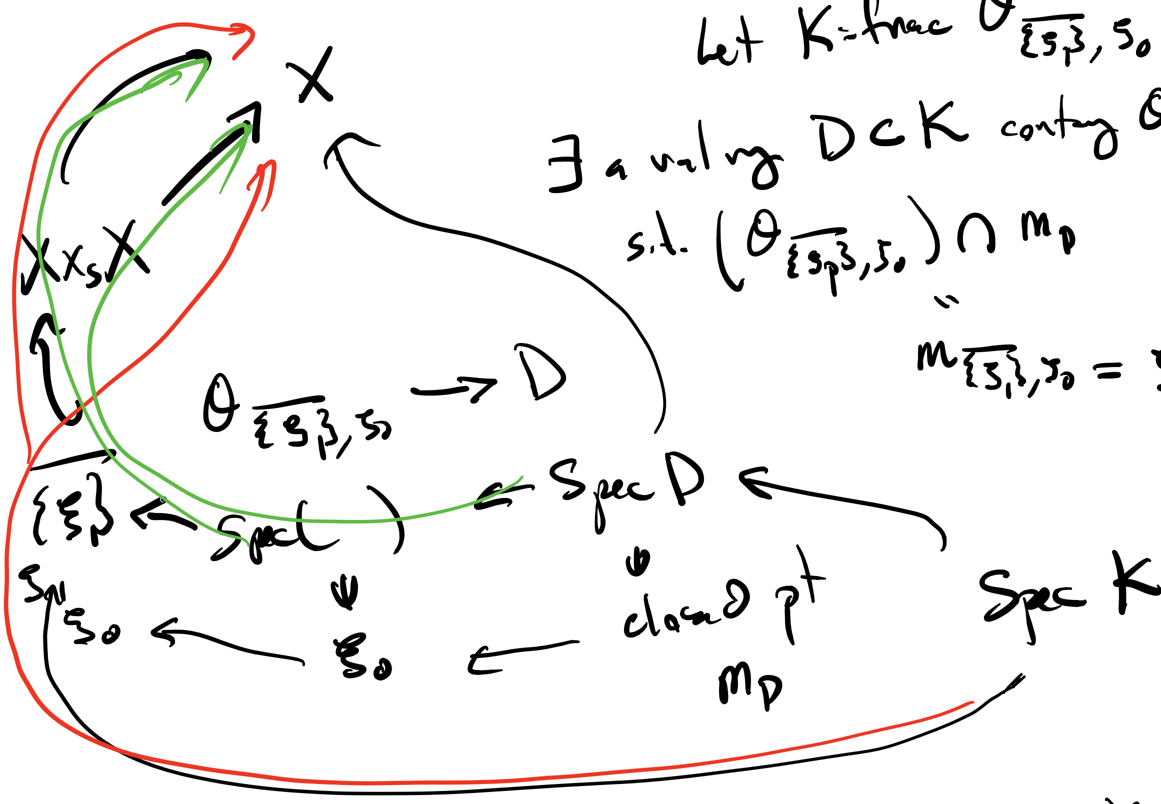
w/ reduced scheme structure

$(R/p)_q$

Let $K = \operatorname{frac} \mathcal{O}_{\overline{\{\xi_1\}},\xi_0}$

$\exists$ a valuation ring $D \subset K$ containing $\mathcal{O}_{\overline{\{\xi_1\}},\xi_0}$

s.t. $\left(\mathcal{O}_{\overline{\{\xi_1\}},\xi_0}\right) \cap m_D$
$\qquad = m_{\overline{\{\xi_1\}},\xi_0} = \xi_0$



$$\operatorname{Spec} K \longrightarrow \operatorname{Spec} k(\xi_1) \hookrightarrow \operatorname{Spec} \mathcal{O}_{\overline{\{\xi_1\}},\xi_0} \to X \times_S X$$

$$\longrightarrow X \xrightarrow{\Delta} X \times_S X$$

$$
\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} D \arrow[r] \arrow[ur, bend left] \arrow[ur, bend right] & S
\end{tikzcd}
$$

agree

$$
\Rightarrow \quad
\begin{tikzcd}
& & X \\
\operatorname{Spec} D \arrow[r] \arrow[urr, bend left, "\text{agree}"] \arrow[drr, bend right, "\text{agree}"'] & X \times_S X \arrow[ur] \arrow[dr] & \\
& & X
\end{tikzcd}
$$

$$
\rightarrow \operatorname{Spec} D \rightarrow X \xrightarrow{\Delta} X \times_S X
$$

$$
m_D \longrightarrow \xi_0
$$

- closed immersions are proper
- composition of proper = proper
- projective space is proper (Noeth base)

Def: $\mathbb{P}^n_X = X \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$

$\mathbb{P}^n_X \longrightarrow X$ is proper (Thm 4.9) if X Noeth

$\Rightarrow$ closed subschemes of proj space are proper

$$\begin{tikzcd} Z \arrow[r, hook, "\text{closed}"] \arrow[dr, "\text{proper}"'] & \mathbb{P}^n_X \arrow[d] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \mathbb{P}^n_{\mathbb{C}} \arrow[d] & Z \arrow[l, hook'] \arrow[dl] \\ \operatorname{Spec}\mathbb{C} & \end{tikzcd}$$