Sheaves of modules

Concrete def (Hartshorne)

Let $(X, \mathcal{O}_X)$ ringed space, a sheaf of $\mathcal{O}_X$-mod's is a sheaf $\mathcal{M}$ on X (of Ab. grps) together w/ maps

$$\mathcal{O}_X(U) \times \mathcal{M}(U) \xrightarrow{\cdot} \mathcal{M}(U) \quad \text{all } U$$

s.t. $\mathcal{M}(U)$ structure of an $\mathcal{O}_X(U)$ module

and s.t. $\forall\, V \subset U$

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{M}(U) & \xrightarrow{\cdot} & \mathcal{M}(U) \\ \downarrow{\scriptstyle res \times res} & & \downarrow{\scriptstyle res} \\ \mathcal{O}_X(V) \times \mathcal{M}(V) & \xrightarrow{\cdot} & \mathcal{M}(V) \end{array}$$

commutes.

Remark / Alternate perspective

If $\mathcal{C}$ any category (w/ products & a terminal object)

then a ring object in $\mathcal{C}$ is an object R

together w/ maps

$$R \times R \xrightarrow{+} R \qquad * \xrightarrow{0} R$$

$$R \times R \xrightarrow{\cdot} R \qquad * \xrightarrow{1} R$$

s.t. a bunch of diagrams commute

$$\begin{tikzcd}
R \times R \times R \arrow[r, "id \times \cdot"] \arrow[d, "\cdot \times id"'] \arrow[dr, phantom, "\circlearrowright"] & R \times R \arrow[d, "\cdot"] \\
R \times R \arrow[r, "\cdot"'] & R
\end{tikzcd}$$

$$\begin{tikzcd}
R = R \times * \arrow[r, "id \times 1"] \arrow[drr, "id"] \arrow[d, equal] & & R \times R \arrow[d, "\cdot"] \\
* \times R \arrow[d, "1 \times id"'] & & R \\
R \times R \arrow[urr, "\cdot"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
R \times R \times R \arrow[rr, "id \times +"] \arrow[d, "\pi_1 \times \pi_2 \times \pi_1 \times \pi_3"'] & & R \times R \arrow[d, "\cdot"] \\
R \times R \times R \times R \arrow[r, "\cdot \times \cdot"'] & R \times R \arrow[r, "+"'] & R
\end{tikzcd}$$

Exercise: a ring object in cat of sheaves on X is a sheaf of rings.

Similarly if R a ring object in $\mathcal{C}$, a R-module object is an object M w/ maps

$$R \times M \xrightarrow{\cdot} M \qquad M \times M \xrightarrow{+} M$$

st . . . .

if we think of $\mathcal{O}_X$ as a ring object in Sh_X ← sheaves of sets on X.
then an $\mathcal{O}_X$-module object = a sheaf of $\mathcal{O}_X$-modules

Def A morphism of sheaves of $\mathcal{O}_X$-modules is "what I just said"

Relatively formal comments:

Can take kernels, cokernels, quotients, images in category of sheaves of $\mathcal{O}_X$-modules

$f: \mathcal{M} \to \mathcal{N}$ $\mathcal{O}_X$-mods

(inherited from Ab sheaves)

Def if $\mathcal{F}$ & $\mathcal{G}$ are $\mathcal{O}_X$-modules then can define a sheaf of $\mathcal{O}_X$ modules

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

"normal" morphisms

Def if $\mathcal{F}$ & $\mathcal{G}$ $\mathcal{O}_X$-modules

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}\ (U) = \cancel{\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)}$$

oops! not a sheaf

correction $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf $U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$

$$\{R\text{-mods}\} \longrightarrow \{\mathcal{O}_{\operatorname{spec} R}\text{-modules}\}$$
$$M \longmapsto \widetilde{M}$$

Def an $\mathcal{O}_X$-module $\mathcal{M}$ is <u>free</u> if $\mathcal{M} \simeq \mathcal{O}_X^N$ some N (possibly infinite)

<u>recall</u>: comm. rings have Invariant basis number (N well defined)

$\Rightarrow N$ stays same under restriction to smaller opens.

if $\mathcal{M}$ is free an $\mathcal{M}(U) = \mathcal{O}_X(U)^N$ then N = rank.

Def an $\mathcal{O}_X$-module $\mathcal{M}$ is locally free if $\exists$ open cover $\mathcal{U}_i$ of X s.t. $\mathcal{M}|_{\mathcal{U}_i}$ is free $\mathcal{O}_X|_{\mathcal{U}_i} = \mathcal{O}_{\mathcal{U}_i}$ module.

($\Rightarrow$ rank of a locally free module is locally constant i.e. if X is connected $\Rightarrow$ rk is constant)

Def An $\mathcal{O}_X$ module is called invertible if it is locally free, rank 1.

Pushforward, Pullback (ex $X = \operatorname{Spec} B$ $Y = \operatorname{Spec} A$, $A \to B$)

Suppose $f: X \longrightarrow Y$ schemes, and $\mathcal{F}$ sheaf on X, $\mathcal{G}$ sheaf on Y

$f_s\mathcal{F}$ $f_*\mathcal{F}$ a sheaf on Y $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ on Y

$f^s\mathcal{G}$ $f^{-1}\mathcal{G}$ a sheaf on X $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ on X

if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$ modules $\Rightarrow$ $f_*\mathcal{F}$ is a sheaf of $f_*\mathcal{O}_X$ mods

$\mathcal{G}$ - — .. $\mathcal{O}_Y$ mods $(Y, f_*\mathcal{O}_X)$ a ringed space

$\Downarrow$

$f^{-1}\mathcal{G}$ is a sheaf of $f^{-1}\mathcal{O}_Y$-mods

$f_*\mathcal{O}_X \times f_*\mathcal{F} \to f_*\mathcal{F}$

$f_*\mathcal{O}_X(U) \times f_*\mathcal{F}(U)$

$\shortparallel$

$\mathcal{O}_X(f^{-1}(U)) \times \mathcal{F}(f^{-1}(U))$

$f_*\mathcal{F}$ is naturally an $\mathcal{O}_Y$-module via

$$\begin{tikzcd}
\mathcal{O}_Y \times f_*\mathcal{F} \arrow[r] \arrow[d, "f^\# \times id"] & f_*\mathcal{F} \\
f_*\mathcal{O}_X \times f_*\mathcal{F} \arrow[ur] &
\end{tikzcd}$$

$\mathcal{F}(f^{-1}(U))$
$\shortparallel$
$f_*\mathcal{F}(U)$

$A \to B$
if M a B-mod
$\Rightarrow$ it's an A-mod

$f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-mod

have a map $f^\#: f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$

Can get an $\mathcal{O}_X$-mod by $\otimes$!

define $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ (via $f^\#$)

$A \to B$
N is an A mod
$\Rightarrow N \otimes_A B$ is a B-mod

Quasi-Coherent sheaves of modules

Def If M is an A-module, we define a sheaf $\tilde{M}$ on $\operatorname{Spec} A$ as follows:

$$\tilde{M}(D_f) = M_f = \left\{ \frac{m}{f^n} \mid m \in M,\ n \in \mathbb{Z}_{\geq 0} \right\}$$

is an R_f-module

$$\frac{m}{f^i} = \frac{n}{f^j} \Leftrightarrow$$

if $D_g \subset D_f$

natural restrictions via $R_f \to R_g$

$$f^k (f^j m - f^i n) = 0$$

some k.

Motivation for Hartshorne def:

want stalks of $\tilde{M}$ to be $\tilde{M}_p = M_p$

$$= \left\{ \frac{m}{f} \mid f \notin p \right\}$$

know $\tilde{M}(U) \to \prod_{p \in U} \tilde{M}_p$

injective

<u>Refine</u> $\tilde{M}(U) = \left\{ \sigma : U \to \bigsqcup_{p \in U} M_p \mid \text{s.t. } p \mapsto \tilde{M}_p \right.$

and for $D_f \subset U$, $\exists\, m \in M_f$

s.t. $\left. \sigma(p) = \text{img. of } m \text{ in } M_p \right\}$

Hartshore $\leftrightarrow$ if $\exists$ nbhd s.t. ... Hartshorne...

then can choose it to be a D_f.

Proposition (5.1)

if M an A-module, $X = \operatorname{Spec} A$ then

- $\widetilde{M}$ an $\mathcal{O}_X$-mod
- $\widetilde{M}_p = M_p$
- $\widetilde{M}(D_f) = M_f$
- $\Gamma(\operatorname{Spec} A, \widetilde{M}) = M$ $\longleftarrow$ parallels computation $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$

Prop (5.2)

The functor $M \longmapsto \widetilde{M}$

$$\{R\text{-mod}\} \longrightarrow \{\mathcal{O}_{\operatorname{Spec} R}\text{-mods}\}$$

is exact, fully faithful and commutes w/ $\otimes$ & $\oplus$

and if $f: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ $\quad (A \to B)$

then $f_*(\widetilde{N}) = \widetilde{{}_A N}$

and $f^*(\widetilde{M}) = \widetilde{M \otimes_A B}$

Def A sheaf of $\mathcal{O}_X$ modules $\mathcal{M}$ on X is quasicoherent if $\exists$ a cover $U_i = \operatorname{Spec} A_i$ of X s.t. $\mathcal{M}|_{U_i} = \widetilde{M}_i$ some A_i-module M_i.

We say $\mathcal{M}$ is coherent if the M_i are finitely generated.

(this is the wrong def. of coherent!)

[Actual def:
$\mathcal{M}$ coherent if $\mathcal{M}$ is loc. finitely generated q.coh
and $\forall U \subset X$ open and morphisms

$$\mathcal{O}_X^N|_U \longrightarrow \mathcal{M}|_U$$

the kernel is loc. f.g. (q.coh)]

Prop (Cor 5.5)

if $X = \operatorname{Spec} A$ then $M \mapsto \widetilde{M}$ is an eq. of cats between

$$\{A\text{-mods}\} \longleftrightarrow \{\text{q.coh. } \mathcal{O}_{\operatorname{Spec} A}\text{-mods}\}$$

Prop (5.6) Γ on q-coh. sheaves on affines is acyclic/exact.

if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact as sheaves of q.c. $\mathcal{O}_X$-mods and X affine

$\Rightarrow 0 \to \Gamma(\mathcal{F}') \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'') \to 0$ exact.

Prop (5.7)
kernels, cokernels, images of morphisms of q.coh. are q.coh.

Prop 5.8 f^* of q.c is q.c. (and coh. is coh.)

f_* of q.c. is q.c. if f is q.c. sep or domain scheme is Noeth.

f_* coh. need not be coherent.