$\mathbb{P}^n = \{ [a_0; \dots; a_n] \mid a_i \in k, \text{ not all } 0 \} = \{ \ell < k^{n+1} \mid \ell \text{ is a line} \}$

$[a_0; \dots; a_n] = [b_0; \dots; b_n]$

if $\exists \lambda \in k^*$ s.t. $a_i = \lambda b_i$ all i

$T = \{ (v, \ell) \mid v \in k^n, \ \ell < k^n, \ \ell \text{ a line} \}$

$= \{ (v_0, \dots, v_n), [a_0; \dots; a_n] \mid a_i = \lambda v_i \text{ some } \lambda \text{ all } i \}$

$$T \xrightarrow{\pi} \mathbb{P}^n$$
$$(v, \ell) \longrightarrow \ell$$

sheaf $\mathcal{M}$ on $\mathbb{P}^n$ via

$\mathcal{M}(U) = \{ f : U \longrightarrow T \mid \pi f = id_U \}$

$\mathcal{M}(U)$ a module over $\mathcal{O}_{\mathbb{P}^n}(U)$

$\mathcal{M}(U) \ni f$, $\quad \mathcal{O}_{\mathbb{P}^n}(U) \ni r$

$f = (f_0(a_0; \dots; a_n), f_1(a_0; \dots; a_n) \dots)$

s.t. $(f_0(\vec{a}), f_1(\vec{a}) \dots f_n(\vec{a})), [a_0; \dots; a_n]$

in T

$T = \{ (v_0, \dots, v_n) [a_0 \dots a_n] \mid \dots \sim \}$

$= \{ (\lambda a_0, \lambda a_1, \dots, \lambda a_n) [a_0; \dots; a_n] \}$

sections determined by λ's functions of a_i's.

want $f_0(\vec{a})\, f_1(\vec{a}) \ldots$

and section given by

$$[a_0; \ldots; a_n] \xrightarrow{f} (f_0(\vec{a})a_0, f_1(\vec{a})a_1, \ldots)$$

$[a_0; \ldots; a_n]$

for this to make sense

$$\ell \longrightarrow (\ell, p)$$

$$\underbrace{f_i(\lambda\vec{a})(\lambda a_i)}_{\text{rat'l func of deg 0}} = f_i(\vec{a})a_i$$

in a's.

$$\frac{a_1+a_2}{a_3}$$

sections of tautological sheaf are in bijection w/ rat'l fns of deg -1.

i.e. on $\mathbb{A}^n_i \subset \mathbb{P}^n$

$\{[a_0; \ldots; a_{i-1}; 1; a_{i+1}; \ldots; a_n]$

coords $\frac{x_j}{x_i}$

then $\frac{1}{x_i}$ is a section of taut. sheaf on this open set.

$f_0\hat{a}_0, f_1\hat{a}_1, \ldots,$

$$f_0(\lambda\vec{a})\,\widehat{\lambda a_0} \rightsquigarrow \lambda^{\deg f_0} f_0(\vec{a})\,\lambda^{-1}\hat{a}_0$$

f_0 deg 1

x_1

Recall: defined coherent, q.coh sheaves of $\mathcal{O}_X$-modules
(only in Noeth case)

if $X = \operatorname{Spec} A$, have an equiv of cats

$$\{\text{q.coh sheaves on } X\} \longleftrightarrow \{A\text{-mods}\}$$
$$\tilde{M} \longleftarrow M$$
$$\mathcal{M} \longrightarrow \Gamma(X, \mathcal{M})$$

if A Noeth $\{\text{coh on } X\} \longleftrightarrow \{\text{f.g } A\text{-mods}\}$

Prop 5.6: if X is affine then $\Gamma : \{\text{q.coh}/X\} \to \{A\text{-mod}\}$ is exact.

Prop 5.7: For a general scheme. kernels, cokernels, images of morphisms of q.coh sheaves are q.coh. Also extensions of q.coh are q.coh. (Also all same for coherent)

Pf of extension part: if $\mathcal{F}', \mathcal{F}''$ are q.coh $\mathcal{O}_X$-mods and

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \text{ is exact.}$$

WTS $\mathcal{F}$ is q.coh. $\mathcal{F}$ q.coh if $\mathcal{F}|_{\operatorname{Spec} A}$ is q.coh.

so WLOG, restrict to case $X = \operatorname{Spec} A$

on $\operatorname{Spec} A = X$,

set $M = \Gamma(X, \mathcal{F})$

$M' \; \cdots \; \mathcal{F}'$

$M'' \; \cdots \; \mathcal{F}''$

by Ex 5.3 $\exists$ nat. maps
$$\tilde{M}' \xrightarrow{\sim} \mathcal{F}'$$
$$\tilde{M} \to \mathcal{F}$$
$$\tilde{M}'' \xrightarrow{\sim} \mathcal{F}''$$

subtext: (prop 5.4)

- $\mathcal{F}$ is q.coh $\iff$ q.coh on an open cover
- $\mathcal{F}$ q.coh on an affine $\iff$ $\mathcal{F} \simeq \tilde{M}$

$$\begin{tikzcd}
Shv(X) \arrow[r, "\Gamma_U \text{ not exact}"] \arrow[rr, "\text{stalk at } p"'] \arrow[d] & Ab.\,gps & Ab\, gp \text{ is exact} \\
Shv(U) & &
\end{tikzcd}$$

get maps

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r, "S"] & M'' \arrow[r] & 0 \quad \boxed{5.6} \\
0 \arrow[r] & \tilde{M}' \arrow[r] & \tilde{M} \arrow[r] & \tilde{M}'' \arrow[r] & 0
\end{tikzcd}$$

S

$$\begin{tikzcd}
0 \arrow[r] & \tilde{M}' \arrow[r] \arrow[d, "\wr"] & \tilde{M} \arrow[r] \arrow[d] & \tilde{M}'' \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0
\end{tikzcd}$$

Actual 5.6: if $X = \operatorname{Spec} A$, $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact seq of $\mathcal{O}_X$-modules and $\mathcal{F}'$ q-coh then $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'')$ exact (and therefore $0 \to \Gamma(\mathcal{F}') \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'') \to 0$ exact.)

Prop: 5.8 if $f: X \to Y$ a morphism then

- f^* of q.coh is q.coh
- if X, Y noeth, f^* of coh is coh
- if X noeth or f. q.comp & sep then f_* q.coh is q-coh.

when is f_* coh coherent? (correct answer: f proper)

Plan:

- Ideal sheaves
- Twisting sheaves
- coherence of f_* under projective morphisms

Def let Y be a closed subscheme of X, $i : Y \to X$ inclusion define $\mathcal{I}_Y = \ker$ of $i^{\#} : \mathcal{O}_X \to i_* \mathcal{O}_Y$

Prop 5.9 $\exists$ a bijection between q.coherent sheaves of ideals of $\mathcal{O}_X$ and closed subschemes of X.

$$\{\mathcal{I} \subseteq \mathcal{O}_X\}$$

{closed subschemes of X} $\quad Y \longmapsto \mathcal{I}_Y = \ker i^{\#}$

$\operatorname{Spec} \mathcal{O}_X/\mathcal{I} \longleftarrow \mathcal{I}$

(ex. 5.17)

consider $\mathcal{O}_X/\mathcal{I}$

let $Y = \operatorname{supp}(\mathcal{O}_X/\mathcal{I}) = \{p \in X \mid (\mathcal{O}_X/\mathcal{I})_p \neq 0\}$

top. this w/ sheaf of rings $\mathcal{O}_X/\mathcal{I}$ $\qquad (A/I)_p \neq 0 \iff I \subseteq p$

$A \longleftrightarrow \operatorname{Spec} A$ $\quad$ pts primes $\quad$ basic opens $D_f \simeq \operatorname{Spec} A_f$ $\quad \mathcal{O}_{\operatorname{Spec} A}(D_f) = A_f$

$S \longleftrightarrow \operatorname{Proj} S$ $\quad$ pts hom primes (not containing irrelevant ideal $S_{>0}$)

basic open $D_f^+ \simeq \operatorname{Spec} A_{(f)}$

M an A mod $\qquad \tilde{M}$ shf of $\mathcal{O}_{\operatorname{Spec} A}$ mods

$$\tilde{M}(D_f) = M \otimes_A A_f = M_f$$

N a $\mathbb{Z}$-graded S-mod $\qquad \tilde{N}$ shf of $\mathcal{O}_{\operatorname{Proj} S}$ mods

$$\tilde{N}(D_{(f)}) = N_{(f)}$$

$= \{$hom elmts of deg 0 in $N_f\}$ mod over $A_{(f)}$

N_f gen by hom elmts

f hom $\qquad \deg \frac{n}{f^i} = \deg n - i \deg f$

$$\left\{\begin{array}{c}\text{graded}\\ S\text{-mods}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{q.coh sheaves of}\\ \mathcal{O}_{\operatorname{Proj} S}\text{-mods}\end{array}\right\}$$

$$N \longmapsto \tilde{N}$$

$$\Gamma_*(\mathcal{F}) \longleftarrow \mathcal{F}$$

graded mod

"id"

something.

$$\cong$$

k_0 = field of char p $\qquad k = k_0(t)$

$L = k(t^{1/p})$

$\operatorname{Spec} \mathbb{C} = X$
$|$
$\operatorname{Spec} \mathbb{R}$

$\operatorname{Spec} L = X$
$\downarrow$
$\operatorname{Spec} k$

$X_{\bar{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$

$\operatorname{Spec} \; k(t^{1/p}) \otimes_k \bar{k}$

$$k(t^{1/p}) = \frac{k[x]}{x^p - t}$$

$X_{\mathbb{C}} \qquad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \dfrac{\mathbb{C}[x]}{x^2+1} =$

$\sim$

$$\frac{\mathbb{C}[x]}{(x+i)(x-i)} \simeq \frac{\mathbb{C}[x]}{x+i} \oplus \frac{\mathbb{C}[x]}{x-i}$$

$\simeq \mathbb{C} \times \mathbb{C}$

$(0,1)\cdot(1,0) = 0$

chose $u \in \bar{k}$
$u^p = t$

$$k(t^{1/p}) \otimes_k \bar{k} \cong \frac{\bar{k}[x]}{x^p - t}$$

$$\| $$

$$\frac{\bar{k}[x]}{x^p - u^p} = \frac{\bar{k}[x]}{(x-u)^p}$$

$x - u \neq 0$

$(x-u)^p = 0$