Algebraic Geometry 2½ supplimentary worksheet 1

Sheaves

Critical Hartshorne problems in II.1.1

- (short term) 1.1, 1.2, 1.3, 1.4, 1.5, 1.18, 1.22
- (for later in semester) 1.8, 1.14, 1.15, 1.17, 1.19, 1.20

Other fun problems below:

1. Suppose we change the definition of a presheaf of sets by dropping condition (0). This makes a presheaf of sets the same as a functor (according to many texts). Does sheafification still work?

   That is, if we define, for a functor $\mathscr{F} : Op(X)^{op} \to Sets$, a new functor $\mathscr{F}^+$ as in Proposition-Definition 1.2, is the resulting object a sheaf with the same universal property?

2. Recall that for sets S, T, and maps $f, g : S \to T$, we define the equalizer $Eq(f, g)$ to be the subset of S consisting of those elements s such that $f(s) = g(s)$.

   For a functor $\mathscr{F} : Op(X)^{op} \to Sets$ on the open sets of a topological space X, if U is open in X, and $\{U_i\}$ are an open covering of U, we define

   $H^0(\{U_i\}, \mathscr{F})$ as the equalizer of the maps

   $$\pi_1, \pi_2 : \prod_i \mathscr{F}(U_i) \to \prod_{(i,j)} \mathscr{F}(U_i \cap U_j)$$

   where π_1 takes a tuple $(s_i)_i$ to the tuple $(t_{i,j})_{i,j}$ where $t_{i,j} = s_i$, and π_2 takes $(s_i)_i$ to $(u_{i,j})_{i,j}$ where $u_{i,j} = s_j$.

   (a) Show that there is a natural map $\mathscr{F}(U) \to H^0(\{U_i\}, \mathscr{F}$. In other words, both sides describe functors which map $\mathscr{F}$ to a set, i.e. functors

   $$Fun(Op(X)^{op}, Set) \to Set,$$

   where the left hand side is evaluation at U. Construct a natural transormation between these sides, induced by restriction maps.

   (b) Show that if $\mathscr{F}$ is a sheaf, then $\mathscr{F}(U) \to H^0(\{U_i\}, \mathscr{F})$ is an isomorphism.

3. Suppose we have covers $\mathscr{U} = \{U_i\}_{i\in I}$ and $\mathscr{V} = \{V_j\}_{j\in J}$ of the same open set $U \subset X$. We define a morphism (aka a refinement map) of covers, to be a map of the index sets $f : I \to J$ such that $U_i \subset V_{f(i)}$ for each i. Show that if f is a refinement as above, we have an induced natural transformation $H^0_{\mathscr{V}} \to H^0_{\mathscr{U}}$ of (covariant) functors from functors to sets, which is an isomorphism for sheaves.

4. Define $H^0(\mathscr{F})(U) = H^0(U, \mathscr{F}) = \lim_{\{U_i\}} H^0(\{U_i\}, \mathscr{F})$ (this is a direct limit, aka a kind of colimit).
   (a) Show that for $\mathscr{F}$ a functor, $H^0(\mathscr{F})$ is a separated presheaf.

   (b) Let X be a topological space with a single point $*$. Let $\mathscr{F}$ be the functor with $\mathscr{F}(\emptyset) = \mathbb{Z}$ and $\mathscr{F}(X) = \{0\}$, with restriction given by the inclusion. Show that $H^0(\mathscr{F})$ is not a sheaf.

   (c) Show that if $\mathscr{F}$ is a presheaf, then $H^0(\mathscr{F})$ is its sheafification.

   (d) Show that for $\mathscr{F}$ a functor, $H^0(H^0(\mathscr{F}))$ is a sheaf, and the map $\mathscr{F} \to H^0(H^0(\mathscr{F}))$ is universal for maps from $\mathscr{F}$ to sheaves.

5. We define $\mathbb{RP}^1$ to be the set of lines in a 2-dimensional real vector space. We endow this with the structure of a topological space as follows: each line intersects the unit circle S^1 in two antipodal points $x, -x$. We may therefore identify $\mathbb{RP}^1 = S^1/\sim$ where $x \sim y$ if and only if $x = \pm y$, and $\mathbb{RP}^1$ is given the quotient topology.

   Let $\mathscr{T} = \{(\ell, v) \in \mathbb{RP}^1 | v \in \ell\}$. This is called the tautological bundle, and it comes with a map $\mathscr{T} \to \mathbb{RP}^1$ via $(\ell, v) \mapsto v$, which then corresponds to a sheaf of vector spaces via considering its sections. We will see later that this is a very important sheaf, so it is a good idea to get a visual/geometric feeling for it.

   (a) Show that $\mathscr{T}$ is homeomorphic to the open Möbeus band $(-1, 1) \times [0, 1]/\sim$ where $\sim$ identifies $(a, 0)$ with $(-a, 1)$.

   (b) Show that every section $s : \mathbb{RP}^1 \to \mathscr{T}$ must satisfy $s(\ell) = (\ell, 0)$ for some ℓ (every section must pass through 0).