Lecture 1: Introduction

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Algebraic Spaces & Stacks
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http://dkrashen.github.io/stacks

What are stacks?
What are algebraic stacks?

Stack ⇝ generalization of a sheaf
Algebraic Stacks ⇝ generalization of a scheme.

Spaces / Algebraic Spaces
Sheaves

Stacks

Alg. Stack

What are all the closed subschemes of $\mathbb{P}^n$?
is a scheme. not f. type, Noeth. — locally f. type
loc. Noeth.

<u>Example</u>: parametrize elliptic curves $/\mathbb{C}$

<u>Recall</u>: elliptic curve is a genus 1 curve E, together w/ a closed pt $e \in E$.

$$y^2 = x(x-1)(x-\lambda) \qquad y^2 z - x(x-z)(x-z\lambda)$$

$$\mathcal{E} = \{((x,y,z), \lambda) \in \mathbb{P}^2 \times (\mathbb{A}^1 \setminus \{0,1\}) \mid$$

x,y,z on above curve for λ }

$$\Big\downarrow$$

$$\mathbb{A}^1 \setminus \{0,1\} = \operatorname{Spec} \mathbb{C}[\lambda]_{\lambda(\lambda-1)}$$

<u>Problem:</u> different λ's give same curve!

$$\lambda \longleftrightarrow \frac{1}{\lambda} \qquad\qquad \lambda \longmapsto \frac{1}{1-\lambda}$$

$$\underbrace{\text{ord } 2 \qquad\qquad\qquad \text{ord } 3}_{S_3 \text{ on } \mathbb{A}^1 \setminus \{0,1\}}$$

Can form quotient $(\mathbb{A}^1 \setminus \{0,1\})/S_3$

fcns on quot. are fcns on $\mathbb{A}^1 \setminus \{0,1\}$ which are invariant under the action of the gp

i.e. $\lambda + \frac{1}{\lambda} + \frac{1}{1-\lambda} + \cdots$

$$j = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} \qquad \mathbb{C}[\lambda]_{\lambda(\lambda-1)}^{S_3} = \mathbb{C}[j]$$

$\mathbb{A}^1_j$ pts are in bijection w/ iso. classes of elliptic curves / $\mathbb{C}$

but: family of curves $\begin{matrix} \tilde{E} \\ \downarrow \pi \\ B \end{matrix}$ $\rightsquigarrow$ give a map $B \to \mathbb{A}^1_j$

$b \mapsto j$ for $\pi^{-1} b$

& different families can give same map!!

$$y^2 z = x^3 - t z^3 \qquad j \equiv 0 \qquad y^2 z = x^3 - z^3$$

t-family   const. t-family

Can show – different elliptic curves over generic pt of t-line!
(over $\mathbb{C}(t)$ different!)

There is no scheme which can do this! i.e. can keep track of families.

Note – these curves give two pts of $\mathbb{A}^1_j$ defined over $\mathbb{C}(t)$
i.e. coincide over $\overline{\mathbb{C}(t)}$ (same j invariant)
$\Longrightarrow$ should force the curves to be $\cong$, but they aren't.

But, we'll see $\exists\ \mathcal{M}_{1,1}$ algebraic stack (smooth Deligne-Mumford)

$$\mathcal{M}_{1,1} \longrightarrow M_{1,1} = \mathbb{A}^1_j \quad \text{coarse moduli space.}$$

Schemes as functors (Yoneda embedding)

When we learned sheaves:

Presheaf on X functor $\quad \underline{Op(X)} \xrightarrow{\mathcal{F}} \underline{Sets}$ (contravariant)

$$U \mapsto \mathcal{F}(U) \qquad U \subset V \quad \mathcal{F}(V) \to \mathcal{F}(U)$$

$\underline{Op(X)}$ some subcategory of $\underline{Sch}$

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$$\underline{Sch/B} \longrightarrow \underline{Sets}$$

$\underline{ex}$: $\mathcal{O} : \begin{matrix} X \\ \downarrow \\ B \end{matrix} \rightsquigarrow \mathcal{O}_X(X)$ $\qquad \mathcal{O}^* : X \longrightarrow \mathcal{O}_X(X)^*$

$$\mathrm{Pic} : \begin{pmatrix} X \\ \downarrow \\ B \end{pmatrix} \longmapsto \mathrm{Pic}\, X$$

$\underline{ex}$: Given a scheme $\begin{matrix} Y \\ \downarrow \\ B \end{matrix}$, define $h_Y : \underline{Sch/B} \to \underline{Sets}$

$h_Y(X) = \mathrm{Mor}_B(X, Y)$ "functor of points" of Y

families of pts in Y, parametrized by X

$\underline{ex}$: $h_{\mathbb{A}^1}(X) = \mathrm{Mor}(X, \mathbb{A}^1) = \mathcal{O}_X(X)$

means natural iso. of functors.

$$h_{\mathbb{A}^1 \setminus \{0\}}(X) = \mathcal{O}_X(X)^*$$

$\mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m \qquad \mathbb{A}^1 = \mathbb{G}_a$

$\underline{\text{Next}}$: Props of morphisms – smooth, étale, f.presented.