Lecture 10: descent 1.5

Thursday, September 18, 2014 11:04 AM

fppf descent:

$f: A \to B$ (faithfully flat morphism)

M A-module $M_B = M \otimes_A B$

$M_{B\otimes 2} = M \otimes_A (B \otimes_A B)$

⋮

$X \to \operatorname{Spec} A$ $\coprod U_i = \operatorname{Spec} B$ $\{U_i \to X\}$

$B \otimes_A B$ "$= \coprod U_i \cap U_j$" M_B "$= \prod M|_{U_i}$"

$M_{B\otimes 2}$ "$= \prod M|_{U_i \cap U_j}$"

Prop If $A \xrightarrow{f} B$ f.flat $\Longrightarrow$ if M an A-module

$$\begin{tikzcd} M \arrow[r, "f"] & M_B \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & M_{B\otimes 2} \end{tikzcd}$$ exact

$$\left(M(X) \to \prod M(U_i) \rightrightarrows \prod M(U_i \cap U_j) \right)$$

Pf: $\otimes B$ faithfully flat! So WLOG can tensor all w/ B:

$$\begin{tikzcd} M_B \arrow[r, "f'"] & M_{B\otimes 2} \arrow[r, shift left, "p_1'"] \arrow[r, shift right, "p_2'"'] & M_{B\otimes 3} \end{tikzcd}$$

$$m\otimes b \longmapsto m\otimes 1\otimes b$$

$$\begin{tikzcd} m\otimes b_1\otimes b_2 \arrow[r, "p_1'"] \arrow[rd, bend right, "p_2'"'] & m\otimes b_1\otimes 1\otimes b_2 \\ & m\otimes 1\otimes b_1\otimes b_2 \end{tikzcd}$$

define $M_{B\otimes 2} \xrightarrow{\eta} M_B$ η section $f' \Rightarrow$

$m\otimes b_1\otimes b_2 \longrightarrow m\otimes b_1 b_2$ f' inj

$$M_{B\otimes 3} \xrightarrow{\tau} M_{B\otimes 2}$$

$$m\otimes b_1\otimes b_2\otimes b_3 \longrightarrow m\otimes b_1\otimes b_2 b_3$$

$\tau p_1' = id$ $\tau p_2'(\alpha) \overset{\shortparallel}{} f'\eta(\alpha)$

suppose have $\alpha\in M_{B\otimes 2}$ s.t.

$p_1'\alpha = p_2'\alpha$ want: $\alpha = f'(\beta)$

$$\begin{array}{ccc} \tau p_1'\alpha & = & \tau p_2'\alpha \\ \shortparallel & & \shortparallel \\ \alpha & & f'(\underbrace{\eta(\alpha)}_{\beta}) \end{array}$$

$\square$

<u>Cor</u>: If $V \longrightarrow U$ f.flat morphism of affine schemes

& X an affine scheme then:

$$h_X(U) \longrightarrow h_X(V) \rightrightarrows h_X(V \times_U V) \quad \text{is exact.}$$

<u>Pf:</u> $U = \operatorname{Spec} A \quad V = \operatorname{Spec} B \qquad X = \operatorname{Spec} R$

$$A \to B \rightrightarrows B^{\otimes 2} \quad \text{exact}$$

$\operatorname{Hom}(R, -)$ into this.

exactness in middle: if $R \xrightarrow{f} B$ s.t.

$$\begin{tikzcd} R \arrow[r, "f"] & B \arrow[r, shift left] \arrow[r, shift right] & B^{\otimes 2} \end{tikzcd}$$

equal then

$$\forall r \in R, \ f(r) \in B \rightrightarrows B^{\otimes 2}$$

$$\Rightarrow f(r) \in A \qquad \square$$

<u>Lem</u> If $F : (Sch)^{op} \to Set$ a presheaf & a big Zariski shf then F is a $\underset{\tau}{\text{fppf}}$ sh.f $\Leftrightarrow \forall$ $\underset{\tau}{\text{f.flat}}$ $V \to U$, loc. f. pres. the sequence

$$F(U) \longrightarrow F(V) \rightrightarrows F(V \times_U V) \quad \text{exact.}$$

<u>Pf:</u> $\{U_i \to U\}_i \rightsquigarrow \{\coprod_i U_i \to U\}$

$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d, equal] & \prod F(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \prod F(U_i \times_U U_j) \arrow[d, equal] \\
F(U) \arrow[r] & F(\coprod U_i) \arrow[r, shift left] \arrow[r, shift right] & F(\coprod U_i \times_U U_j) \\
& \{U_i \to \coprod_j U_j\}_i \ \text{Zariski cov} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd} F(U_i) \arrow[d, equal] \\ \prod F(U_i \times U_i) \end{tikzcd}$$

$$F(\coprod U_j) \to \prod F(U_i) \rightrightarrows \prod F(U_i \times_{\coprod U_{ik}} U_j)$$

$F(\emptyset)$

$$\begin{tikzcd} & U_i \arrow[dr] & \\ \emptyset \arrow[ur] \arrow[dr] & & \coprod U_j \\ & U_k \arrow[ur] & \end{tikzcd}$$

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Lemma If $F : (Sch)^{op} \to Sets$ presheaf s.t.

i) F a big Zariski sheaf

ii) If $V \to U$ fppf morph. of affines, then

$$F(U) \to F(V) \rightrightarrows F(V \times_U V) \text{ exact}$$

$\Rightarrow$ F is an fppf sheaf

Cor If X a scheme $\Rightarrow$ h_X an fppf sheaf

Pf NEXT TIME

Descent: $\coprod U_i \to X$ cover

$$\begin{tikzcd} F \arrow[d] \\ Sch/X \end{tikzcd}$$

Poetry things are good if

$F(X)$ same as $F(\coprod U_i) = \prod F(U_i)$ together w/ gluing data

$$F(U_i)|_{U_i \cap U_j} \xrightarrow[\sigma_{ij}]{\sim} F(U_j)|_{U_i \cap U_j}$$

$|_{U_i \cap U_j}$ σ_{ij} [illegible])

s.t. $\sigma_{jk}|_{U_i \cap U_j \cap U_k} \circ \sigma_{ij}|_{U_i \cap U_j \cap U_k} = \sigma_{ik}|_{U_i \cap U_j \cap U_k}$

$$\begin{tikzcd} F(\coprod U_i \to X) \\ F(X) \arrow[u] \end{tikzcd}$$

(f_i , σ_{ij}) "descent data"

objects ↗ $\overset{\wedge}{F(U_i)}$ ↖ satfy $\sigma_{ij}: f_i|_{ij} \to f_j|_{ij}$

$\sigma_{jk} \circ \sigma_{ij}| = \sigma_{ik}|_{ijk}$

Non poetry, real life:

$$\begin{tikzcd} F \arrow[d, "p"] \\ C \end{tikzcd}$$

fibred cat, C cat w/ fiber products.

pick a morphism $f: X \longrightarrow Y$ in C

choose pullbacks g^* for each g in C

Define $F(X \xrightarrow{f} Y)$ objects (E, σ) where $E \in F(X)$ and where $\sigma: p_1^* E \xrightarrow{\sim} p_2^* E$

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \end{tikzcd}$$

such that

$(U_i \cap U_j)$ (U_i)

$$\begin{tikzcd}
X \times X \times X \arrow[r, shift left=2, "p_{12}"] \arrow[r, "p_{13}"] \arrow[r, shift right=2, "p_{23}"'] & X \times_Y X \arrow[r, shift left=1, "p_1"] \arrow[r, shift right=1, "p_2"'] & X
\end{tikzcd}$$