Lecture 11: descent, part 2

Tuesday, September 23, 2014 11:00 AM

Lemma If $F : (Sch)^{op} \to Sets$ presheaf s.t.

i) F a big Zariski sheaf

ii) If $V \to U$ fppf morph. of affines, then

$$F(U) \to F(V) \rightrightarrows F(V \times_U V) \text{ exact}$$

$\Longrightarrow$ F is an fppf sheaf

Pf: Suppose F satisfies i) & ii)

Step 1: Let $V \xrightarrow{f} U$ is a f.flat morph. of loc. finite presentation. Then $F(U) \hookrightarrow F(V)$

Pf: Let U_i be an affine cover of U, $V_i = f^{-1}(U_i)$ & V_{ij} affine cover of V_i

$$\begin{array}{ccc} F(U) & \xrightarrow{\text{inj?}} & F(V) \\ \downarrow & & \downarrow \\ \prod F(U_i) & \to & \prod F(V_{ij}) \end{array}$$

inj

$$F(U_i) \to \prod F(V_{ij})$$
$$\qquad\qquad = F(\coprod V_{ij}) \rightrightarrows$$

exact from ii)

Next

Next

Step 2: Want to show: $V \xrightarrow{f} U$ fppf cover

$$F(U) \to F(V) \rightrightarrows F(V \times_U V)$$

exact in middle

Reduce to case U affine.

Let U_i affine cov of U, $V_i = f^{-1}(U_i)$ then we have a comm. diagram:

$$\begin{tikzcd}
 & F(U) \arrow[r] \arrow[d] & F(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(V \times_U V) \arrow[d] \\
\text{exact} \arrow[r] & \prod F(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod F(V_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod F(V_i \times_{U_i} V_i) \\
 & \prod F(U_i \cap U_j) \arrow[r] & \prod F(V_i \cap V_j) &
\end{tikzcd}$$

verticals exact by i)

injective from $F(U_i \cap U_j) \to F(V_i \cap V_j)$ (step 1)

So if middle diagrams

$$F(U_i) \to F(V_i) \rightrightarrows F(V_i \times_{U_i} V_i) \quad \text{exact}$$

$\Rightarrow$ diagram chase give exactness on top.

So suffices to check case U affine.

Step 3: $V \xrightarrow{f} U$ cov fppf U affine, want

$$F(U) \to F(V) \rightrightarrows F(V \times_U V) \quad \text{exact.}$$

reduce to case V q-compact.

<u>Claim/exercise</u>: $\exists$ open cover $V = \bigcup V_j$ s.t. each V_j q-compact
Zariski

&, $V_j \to U$ surjective

<u>Observation</u>: Assuming true for V q-compact then:

$$\mathrm{Eq}\left(F(V) \rightrightarrows F(V \times_U V)\right)$$
$$\|$$
$$F(U) = \mathrm{Eq}\left(F(V_i) \rightrightarrows F(V_i \times_U V_i)\right)$$

given $x \in F(V)$ get $x_i \in F(V_i)$

$\mathrm{Eq}(\,)$ $\mathrm{Eq}(\,)$

assume true q-compact, get for each i, a $y_i \in F(U)$

$$\begin{tikzcd} & F(V_i) \ni x_i \arrow[dr] \arrow[drr] & & \\ F(U) \arrow[ur] \arrow[dr] \arrow[ddrr, bend right] & & & F(V_i \times_U V_j) \\ y_i \overset{?}{=} y_j & F(V_j) \ni x_j \arrow[urr] \arrow[dr] & & \\ & & F(V) \ni x \arrow[uur] & \end{tikzcd}$$

$V_i \times_U V_j \to U$ fppf cover.   etcetera.

q-compact, get $V = \bigcup V_i$ affine finite

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d, equal] & F(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & F(V \times_U V) \arrow[d, hook] \\ F(U) \arrow[r] & F(\coprod V_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & F(\coprod V_i \times_U V_j) \\ & \sim & F((\coprod V_i) \times_U (\coprod V_i)) \arrow[u, equal] \end{tikzcd}$$

bottom exact by ii, done by claim. $\square$

Descent: $\{\coprod U_i \xrightarrow{\amalg f_i} X\}$ cov $(\coprod U_i) \times_X (\coprod U_i)$

$= \coprod_{i,j} U_i \times_X U_j$

"$\coprod U_i \cap U_j$"
(a,b) s.t.
$a \in U_i$, $b \in U_j$
$a = b$ in X $\leftarrow$ (a,b), $f_i(a) = f_j(b)$

Setup: have a fibred cat $\begin{matrix} F \\ \downarrow p \\ C \end{matrix}$, C has fiber products, & we've chosen pullback maps.

pick morphism $X \xrightarrow{f} Y$

$F(X \xrightarrow{f} Y)$ "Descent data" objects on X + data to glue to obj on Y.

$\mathrm{Ob}(F(X \xrightarrow{f} Y)) = (E, \sigma)$ $E \in F(X)$ $\sigma_{ij} : E_i|_{U_{ij}} \to E_j|_{U_{ij}}$

$\pi_1, \pi_2 : X \times_Y X \to X$ $\sigma : \pi_1^* E \to \pi_2^* E$

s.t. $X \times_Y X \times_Y X \xrightarrow{\pi_{12}, \pi_{23}, \pi_{13}} X \times_Y X$

$\pi_{23}^* \sigma \circ \pi_{12}^* \sigma = \pi_{13}^* \sigma$

$\sigma_{jk} \circ \sigma_{ij} = \sigma_{ik}$

$U_{ijk} = U_i \cap U_j \cap U_k$

$\pi_{23}^*\sigma$

$$\begin{tikzcd} \pi_1^*E \arrow[d, "\sigma"] & \pi_{23}^*\pi_1^*E \arrow[d, "\pi_{23}^*\sigma"] \\ \pi_2^*E & \pi_{23}^*\pi_1^*E \end{tikzcd}$$

$$\begin{tikzcd} V \times V \times V \arrow[d] \\ V \times V \arrow[d] \\ V \end{tikzcd}$$

$\pi_{12}^*\sigma$

$$\begin{tikzcd} \pi_1^*E \arrow[d, "\sigma"] & \pi_{12}^*\pi_1^*E \arrow[d, "\pi_{12}^*E"] \\ \pi_2^*E & \pi_{12}^*\pi_2^*E \end{tikzcd}$$

$\pi_{12}^*\pi_2^*$ $\quad \pi_{23}^*\pi_1^*$ both pullbacks of

$\pi_{123 \mapsto 2}$

$F(X \xrightarrow{f} Y)$ $\qquad (E,\sigma)$ $\qquad \sigma : \pi_1^*E \to \pi_2^*E$

$\pi_{23}^*\sigma \circ \pi_{12}^*\sigma = \pi_{13}^*\sigma$

Morphism of descent data:

$(E,\sigma) \longrightarrow (E',\sigma')$ is a morphism

$g : E \longrightarrow E'$ in $F(X)$

s.t. commutes w/ gluing data σ

$$\begin{tikzcd} \pi_1^*E \arrow[r, "\pi_1^*g"] \arrow[d, "\sigma"'] & \pi_1^*E' \arrow[d, "\sigma'"] \\ \pi_2^*E \arrow[r, "\pi_2^*g"'] & \pi_2^*E' \end{tikzcd}$$ commutes.

There's a natural functor:

$$\varepsilon : F(Y) \longrightarrow F(X \xrightarrow{f} Y)$$

$$E \longmapsto (f^*E, \sigma) \qquad \sigma : \pi_1^* f^* E \to \pi_2^* f^* E$$

$$\begin{tikzcd} X \times_Y X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

both pullbacks over same arrow in $\mathcal{C}$

so σ is the unique cart. arrow over id.

<u>Def</u>. We say $X \xrightarrow{f} Y$ is an effective descent morphism for F if $\varepsilon : F(Y) \longrightarrow F(X \xrightarrow{f} Y)$ is an equiv.

Similarly, can generalize to collections of morphisms

$\{X_i \xrightarrow{f_i} Y\}_{i \in I}$ Nice case: $F(\coprod X_i) \xrightarrow{\simeq} \prod F(X_i)$

then can reformulate above for

$\{\coprod X_i \to Y\}$

<u>Def</u> Let $\mathcal{C}$ be a site, $F \xrightarrow{\pi} \mathcal{C}$ category fibred in groupoids. We say F is a stack (over $\mathcal{C}$) if for every $X \in \mathcal{C}$ & cover $\{X_i \to X\}$, $\varepsilon : F(X) \to F(\{X_i \to X\})$ is an equiv.