

Lemma If $F: (\text{Schemes})^{\text{op}} \rightarrow \text{Sets}$ presheaf s.t.

i) F a big Zariski sheaf

ii) If $V \rightarrow U$ fppf morph. of affines, then

$$F(U) \rightarrow F(V) \rightrightarrows F(V \times_U V) \text{ exact}$$

$\Rightarrow F$ is an fppf sheaf

Pf: Suppose F satisfies i) & ii)

Step 1: Let $V \xrightarrow{f} U$ is a f.flat morph. of loc. finite presentation. Then $F(U) \hookrightarrow F(V)$

Pf: Let U_i be an affine cover of U , $V_i = f^{-1}(U_i)$

$\hookrightarrow V_{ij}$ affine cover of V_i

$$\begin{array}{ccc}
 F(U) & \xrightarrow{i_1^?} & F(V) \\
 \downarrow & & \downarrow \\
 \text{TF}(U_i) & \rightarrow & \text{TF}(V_{ij}) \\
 \nearrow i_1 & & \text{"} \\
 F(U_i) & \rightarrow & \text{TF}(V_{ij}) \\
 & & \text{"} \\
 & & F(\coprod V_{ij}) \Rightarrow
 \end{array}$$

exact from ii)

Next

\dots f is a flat cover

Next

Step 2:

Want to show:

$$V \xrightarrow{f} U \text{ fppt cover}$$

$$F(U) \rightarrow F(V) \rightrightarrows F(V \times_U V)$$

exact in middle

Reduce to case U affine.

Let U_i affine cover of U , $V_i = f^{-1}(U_i)$ then we

have a comm. diagram:

$$\begin{array}{ccccc}
 F(U) & \longrightarrow & F(V) & \rightrightarrows & F(V \times_U V) \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{exact} \rightarrow \pi F(U_i) & \longrightarrow & \pi F(V_i) & \rightrightarrows & \pi F(V_i \times_{U_i} V_i) \\
 \downarrow \downarrow & & \downarrow \downarrow & & \\
 \pi F(U_i \cap U_j) & \longrightarrow & \pi F(V_i \cap V_j) & &
 \end{array}$$

verticals exact by i)

↑ injective

from $F(U_i \cap U_j) \rightarrow F(V_i \cap V_j)$
(step 1)

So if middle diagrams

$$F(U_i) \rightarrow F(V_i) \rightrightarrows F(V_i \times_{U_i} V_i) \text{ exact}$$

\Rightarrow diagram chase give exactness on top.

so suffices to check case U affine.

Step 3:

$V \xrightarrow{f} U$ cover fppt U affine, want

$$F(U) \rightarrow F(V) \rightrightarrows F(V \times_U V) \text{ exact}$$

reduce to case V q -compact.

Claim/Exercise: \exists open cover $V = \cup V_j$ s.t. each V_j q -compact
Zariski

\exists $V_j \rightarrow U$ surjective

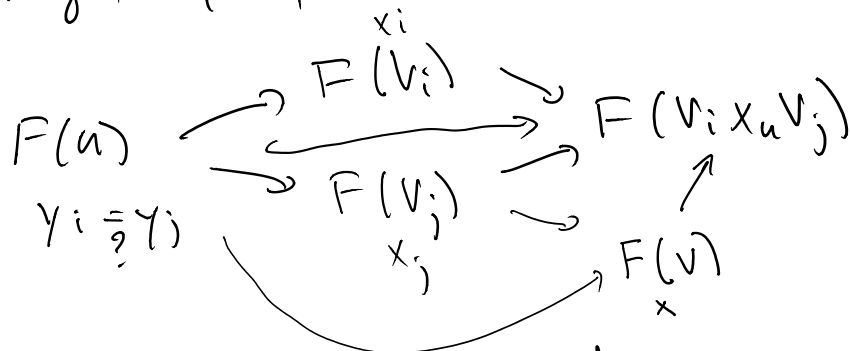
Observation: \rightarrow Assume true for V q -compact then:

$$E_g(F(U) \rightrightarrows F(U \times_U V))$$

$$F(U) = E_g(F(V_i) \rightrightarrows F(V_i \times_U V_i))$$

guess $x \in F(U)$ get $x_i \in F(V_i)$
 $\uparrow E_g(x)$ $\uparrow E_g(\cdot)$

assume true q -compact, get for each i , a $y_i \in F(U)$



$V_i \times_U V_j \rightarrow U$ q -compact. etcetera.

q -compact, get $V = \cup V_i$ of the finite

$$\begin{array}{ccccc} F(U) & \rightarrow & F(V) & \rightrightarrows & F(U \times_U V) \\ \downarrow & & \downarrow & & \downarrow \\ F(U) & \rightarrow & F(\coprod V_i) & \rightrightarrows & F(\coprod V_i \times_U V_j) \\ & & \downarrow & & \downarrow \\ & & \sim & & F((\coprod V_i) \times_U (\coprod V_i)) \end{array}$$

notation read by \cup , done by desc.



Descent: $\{ \coprod U_i \xrightarrow{\cup f_i} X \}$ cover $(\coprod U_i) \times_X (\coprod U_i)$

$$= \coprod_{i,j} U_i \times_X U_j$$

$\leftarrow (a,b), f_i(a) = f_j(b)$
 $\cup U_i \cap U_j$
 $(a,b) \text{ s.t.}$
 $a \in U_i, b \in U_j$
 $a = b \text{ in } X$

Setup: have a fibred cat \mathcal{F} over \mathcal{C} , \mathcal{C} has fiber products, \mathcal{F} has fiber products, if we've chosen pullback maps.

pick morphism $X \xrightarrow{f} Y$

$\mathcal{F}(X \xrightarrow{f} Y)$ "Descent data" objects on $X \rightarrow$ data to glue to obj on Y .

$$\text{Ob}(\mathcal{F}(X \xrightarrow{f} Y)) = (E, \sigma) \quad E \in \mathcal{F}(X)$$

$$\sigma_{ij}: E_i \rightarrow E_j \text{ on } U_{ij}$$

$$\pi_1, \pi_2: X \times_Y X \rightarrow X \quad \sigma: \pi_1^* E \rightarrow \pi_2^* E$$

s.t.

$$X \times_Y X \times_Y X \xrightarrow{\pi_{12}, \pi_{13}, \pi_{23}} X \times_Y X$$

$$\pi_{23}^* \sigma \circ \pi_{12}^* \sigma = \pi_{13}^* \sigma$$

$$\sigma_{jk} \circ \sigma_{ij} = \sigma_{ik}$$

$$U_{i,j,k} = U_i \cap U_j \cap U_k$$

$$\pi_{23}^* \sigma$$

$$\begin{array}{c} \pi_1^* E \\ \downarrow \sigma \\ \pi_2^* E \end{array}$$

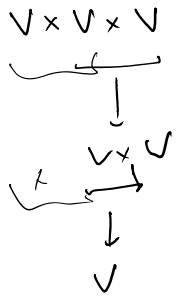
$$\begin{array}{c} \pi_{23}^* \pi_1^* E \\ \downarrow \pi_{23}^* \sigma \\ \pi_{23}^* \pi_2^* E \end{array}$$



$$\pi_{12}^* \sigma$$

$$\begin{array}{c} \pi_1^* E \\ \downarrow \sigma \\ \pi_2^* E \end{array}$$

$$\begin{array}{c} \pi_{12}^* \pi_1^* E \\ \downarrow \pi_{12}^* \sigma \\ \pi_{12}^* \pi_2^* E \end{array}$$



$\pi_{12}^* \pi_2^*$ $\pi_{23}^* \pi_1^*$ both pullbacks of

$$\pi_{123} \mapsto 2$$

$$F(X \xrightarrow{f} Y)$$

$$(E, \sigma)$$

$$\sigma: \pi_1^* E \rightarrow \pi_2^* E$$

$$\pi_{23}^* \sigma \pi_{12}^* \sigma = \pi_{13}^* \sigma$$

Morphism of descent data:

$$(E, \sigma) \longrightarrow (E', \sigma') \text{ is a morphism}$$

$$g_i: E \rightarrow E' \text{ in } F(X)$$

s.t. commutes w/ aty data σ

$$\begin{array}{ccc} \pi_1^* E & \xrightarrow{\pi_1^* \sigma} & \pi_1^* E' \\ \downarrow \sigma & & \downarrow \sigma' \end{array}$$

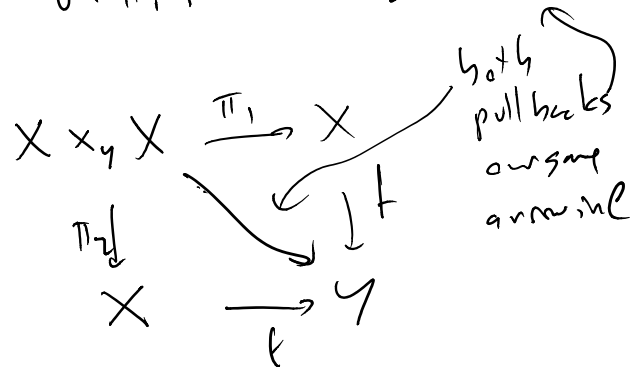
$$\begin{array}{ccc} \pi_2^* E & \xrightarrow{\pi_2^* \sigma} & \pi_2^* E' \\ & & \text{commutes.} \end{array}$$

There's a natural functri

$$\varepsilon: F(Y) \longrightarrow F(X \xrightarrow{f} Y)$$

$$E \longmapsto (f^*E, \sigma)$$

$$\sigma: \pi_1^* f^* E \rightarrow \pi_2^* f^* E$$



So σ is the unique cart. arrow as id.

Def. We say $X \xrightarrow{f} Y$ is an effective descent morphism for F if $\varepsilon: F(Y) \longrightarrow F(X \xrightarrow{f} Y)$ is an equiv.

Similarly, can generalize to collections of morphisms

$$\left\{ X_i \xrightarrow{f_i} Y \right\}_{i \in I} \quad \text{Nice case: } F(\coprod X_i) \cong \prod F(X_i)$$

then can reformulate above for

$$\left\{ \coprod X_i \rightarrow Y \right\}$$

Def Let \mathcal{C} be a site, $F \rightarrow \mathcal{C}$ category fibred in groupoids. we say F is a stack over \mathcal{C} if for every $X \in \mathcal{C}$ & any $\{X_i \rightarrow X\}$, $\varepsilon: F(X) \longrightarrow F(\{X_i \rightarrow X\})$ is an equiv.