Lecture 12: descent, part 3

Thursday, September 25, 2014 11:02 AM

Highlights from descent theory

<u>Recall</u>: $\begin{array}{c} F \\ \downarrow p \\ C \end{array}$ fibred cat, $\mathcal{C}$ category

$F(\{X_i \to Y\})$ objects: $(\{E_i\}, \{\sigma_{ij}\})$

$E_i \in F(X_i)$

$$\begin{tikzcd} X_i \times_Y X_j \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X_i \arrow[d] \\ X_j \arrow[r] & Y \end{tikzcd}$$

$\sigma_{ij} : \pi_1^* E_i \to \pi_2^* E_j$

s.t. "$\sigma_{jk}\sigma_{ij} = \sigma_{ik}$"

$(\pi_{23}^* \sigma_{jk})(\pi_{12}^* \sigma_{ij}) = \pi_{13}^* \sigma_{ik}$

in $F(X_i \times_Y X_j \times_Y X_k)$

or $F(X \xrightarrow{f} Y)$ objects: (E, σ) $\sigma : \pi_1^* E \to \pi_2^* E$

$(\pi_{23}^* \sigma)(\pi_{12}^* \sigma) = \pi_{13}^* \sigma$

<u>Lem.</u> if coproducts exist in $\mathcal{C}$ & if $F(\coprod X_i) = \prod F(X_i)$

then $\{X_i \to Y\}$ effective for $F \iff \coprod X_i \to Y$ is effective.

$\varepsilon : F(Y) \longrightarrow F(X \to Y)$

$\longrightarrow F(\{X_i \to Y\})$

Examples

Prop If $X \xrightarrow{f} Y$ has a section $s: Y \to X$ then $X \to Y$ effective.

Pf: Idea:

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[rr, dashed, bend left, "\simeq \text{id.}"] & F(X \to Y) \arrow[r] & F(Y) \\ & (E, \sigma) \arrow[r] & s^* E \end{tikzcd}$$

for other direction: need to show

$$F(X \to Y) \to F(Y) \xrightarrow{\varepsilon} F(X \to Y)$$

$$(E, \sigma) \to (s^* E) \to (f^* s^* E, \sigma_{\text{canonical}})$$

$\sim ?$

$\square$

Ex: Sheaves

$\mathcal{C}$ site, define a fibred category

$$\begin{array}{c} Sh \\ \downarrow \\ \mathcal{C} \end{array}$$

via $(\mathcal{F}, X)$, $\mathcal{F} \in \widetilde{\mathcal{C}/X}$

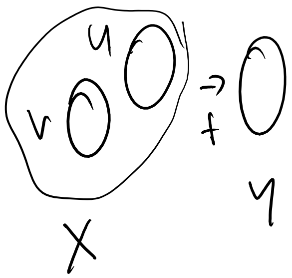
$(\mathcal{G}, X') \to (\mathcal{F}, X)$ is $f: X' \to X$ & $\mathcal{G} \to f^* \mathcal{F}$

for $X \in \mathcal{C}$, $Sh(X) = \widetilde{\mathcal{C}/X}$

Theorem if $f: X \to Y$ a cov in $\mathcal{C}$, then f is an effective desent morphism for Sh.

Pf: $Sh(Y) \xrightarrow{\varepsilon} Sh(X \to Y)$ fully faithful not bad trick is essential surj.

idea of essent. surj (gluing sheaves)



$\mathcal{F}_U/U$ $\qquad$ $f_*\mathcal{F}_U \oplus f_*\mathcal{F}_V$

$\mathcal{F}_V/V$ $\qquad$ then identify

$f_*\mathcal{F}_{U\cap V}$

$\mathcal{F} \equiv$ equalizer$\left(f_*\mathcal{F}_U \oplus f_*\mathcal{F}_V \rightrightarrows f_*\mathcal{F}_{U\cap V}\right)$

more formally, given (E,σ) $\qquad E \in Sh(X)$

$$f_*E \underset{``\pi_1^*"}{\overset{``\sigma\pi_2^*"}{\rightrightarrows}} g_*\pi_1^*E$$

equalizer gives a sheaf mapping to something isom to (E,σ) via ε.

$f: X \to Y$

$$\begin{tikzcd} X\times_Y X \arrow[r, "g"] \arrow[d, "\pi_1"] & Y \\ X & \end{tikzcd}$$

$\square$

<u>Cor</u> If X, Y are schemes $/S$ & $S' \to S$ is fppf cover,

$X' = X_{S'}$ $\quad X'' = X_{S'\times S'}$ $\quad Y' = \dots$ $\quad Y'' = \dots$

<u>then</u>:

- A morphism $f: X \to Y$ is uniquely determined by its pullback $f': X' \to Y'$
- If $f': X' \to Y'$ an S'-morphism such that

$\pi_2^*f', \pi_1^*f' : X'' \to Y''$ $\qquad S'\times_S S' \xrightarrow{\pi_1,\pi_2} S'$

$\pi_1^*f' = \pi_2^*f'$, then $f' = f_{S'}$ some f

$\pi_2^*\ , \pi_1^* \ldots$

$\longrightarrow \pi_1^* f' = \pi_2^* f'$, then $f' = f_{S'}$ same f

Pf: Sketch: Since h_X & h_Y are fppf sheaves, we have

$$\mathrm{Hom}_S(X,Y) = \mathrm{Hom}_{Sh(S)}(h_X, h_Y) = \mathrm{Hom}_{Sh(S' \to S)}((h_{X'}, \sigma_{can}), (h_{Y'}, \sigma_{can}))$$

$$\{ f' : X' \to Y' \text{ s.t. } \pi_1^* f' = \pi_2^* f' \}$$

$\square$

Stuff for Sh works for modules too:

If $\mathcal{C}$ a site, $\mathcal{O}$ a sheaf of rings on $\mathcal{C}$

get for $X \in \mathcal{C}$, a sheaf of rings $\mathcal{O}_X$ on $\mathcal{C}/X$ &

Mod_X = corresp. cat of modules ($\mathcal{O}_X$-modules)

gives a fibred cat
$$\begin{array}{c} MOD \\ \downarrow \\ \mathcal{C} \end{array} \qquad MOD(X) = \mathrm{Mod}_X$$

given $(X, E) \in MOD(X) \subset MOD$

(Y, F), a morphism $(X, E) \longrightarrow (Y, F)$ is a pair (f, v)

$f : X \to Y$

$v : E \to f^* F \in \mathrm{Mod}_X$

Thm if $X \to Y$ a cov in $\mathcal{C}$ then

$\mathrm{Mod}_Y \xrightarrow{\sim} MOD(X \to Y)$ (effec. desc.)

Quasi Coherent big sheaves

S scheme, $\mathcal{O}$ natural presheaf of rngs on S_{fppf} defined by

for an S-scheme T, $\mathcal{O}(T) = \Gamma(T, \mathcal{O}_T)$

this is an fppf sheaf. (because $\mathcal{O}(T) = Hom_S(T, \mathbb{A}^1_S)$)

$Qcoh(S)$ = cat of q-coherent sheaves

given $F \in Qcoh(S)$, can obtain a sheaf F_{big} of $\mathcal{O}$-modules

via for $T \xrightarrow{f} S$, $F_{big}(T) = \Gamma(T, f^*F)$

Lemma If F is q.coh. then F_{big} is a sheaf (fppf)

Pf: know: ets sheaf for big Zariski, for fppf affine covers.

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