Lecture 13: Examples of Descent

Tuesday, September 30, 2014 11:01 AM

Quasi-Coherent shvs. / S

Pick for each $f: X \to Y$ S-schemes pullbacks f^*

Given $F \in QCoh(Y)$ choose f^*'s so that if $S' \hookrightarrow S$ open subscheme

$f^*F = F|_{S'}$

Given $F \in QCoh(S)$ define (pre)sheaf on S_{fppf}

via $F_{big}(T \xrightarrow{f} S) = \Gamma(T, f^*F)$ sheaf of $\mathcal{O}$-mods.

Lem F_{big} is a sheaf on S_{fppf}

Pf: • F_{big} is Zariski sheaf ✓

• descent for affine fppf's: $\operatorname{Spec} B \to \operatorname{Spec} A$, $A \hookrightarrow B$ f.flat

i.e. we have an exact seq.

$$0 \to M \to M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B \qquad \text{D}$$

Notation: if F a sheaf on S_{fppf}, F_S for restriction to Zariski on S

Rem $(F_{big})_S = F$

Lemma G any sheaf of $\mathcal{O}$-mods on S_{fppf} & F a q.coh. sheaf of $\mathcal{O}_S$-mods, then

$$\mathrm{Hom}_{\mathcal{O}}(F_{big}, G) = \mathrm{Hom}_{\mathcal{O}_S}(F, G_S)$$

Pf: Both sides are global sections of Zariski sheaves on S.
E to look locally on S, in particular WLOG

$$F_1 \to F_0 \to F \to 0 \quad F_i \text{ free } \mathcal{O}_S^I$$

$$\begin{array}{ccccccc}
0 \to & \mathrm{Hom}(F, G_S) & \to & \mathrm{Hom}(F_0, G_S) & \to & \mathrm{Hom}(F_1, G_S) & \\
& \uparrow & & \uparrow & & \uparrow & (\text{restriction}) \\
0 \to & \mathrm{Hom}(F_{big}, G) & \to & \mathrm{Hom}(F_{0\,big}, G) & \to & \mathrm{Hom}(F_{1\,big}, G) &
\end{array}$$

et check true if F free (reduce to F_0, F_1)

$\rightsquigarrow$ enough to check $F = \mathcal{O}_S$

$$\mathrm{Hom}(\mathcal{O}, G) \quad \text{vs.} \quad \mathrm{Hom}(\mathcal{O}_S, G_S)$$

$$G(S) = G(S) \qquad \square.$$

Def A big q.coh. sheaf on S is a sheaf on S_{fppf} of the form

F_{big}, F q.coh on S.

Alt: for every $T \to S$, $G|_{T_{Zariski}}$ "G_T" is q.coh sheaf

for every $T' \xrightarrow{g} T$, the map $g^* G_T \xrightarrow{\sim} G_{T'}$

Prop:
$$\begin{tikzcd} QCoh(S) \arrow[r, bend left, leftarrow, "G_S \quad G"] \arrow[r, bend right, "F \quad F_{big}"'] & QCoh(S_{fppf}) \end{tikzcd}$$
big QCoh

old standard   eq. of cats

Theorem
$$\begin{array}{c} QCOH \\ \downarrow \\ S_{fppf} \end{array}$$
has effective descent.

$QCoh(T_{fppf})$

$\wr$

$T \to S$

Pf? Idea: Want to show we have an equiv. of cats

$$QCOH(Y) \xrightarrow{\sim} QCOH(X \xrightarrow{f} Y) \quad f, \text{ fppf cover.}$$

if f is q.compact q.sep then old proof works.

point: pushforward of q.coh. over q.compact morph is q.coh.

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in general, cover Y by affines Y_i, $X_i = f^{-1} Y_i$

cover X_i by q.compacts X_{ij} s.t. X_{ij} cover of Y_i

q.sep.

general structure of argument: use above for $X_{ij} \to Y_i$

rest is Zariski gluing.. $\square$

Similarly, can define $QCoh(S_{\acute{e}t})$, we get

$$\begin{array}{c} QCOH_{\acute{e}t} \\ \downarrow \\ S_{\acute{e}t} \end{array}$$

has descent for étale covers

Similarly, $Coh(S_{\acute{e}t})$ if S loc. Noeth.

$$\begin{array}{c} COH_{\acute{e}t} \\ \downarrow \\ S_{\acute{e}t} \end{array}$$

has étale descent

Closed subschemes

$$\begin{array}{c} CLOSED \\ \downarrow \\ S_{fppf} \end{array}$$ has fppf descent.

$$CL(X) = \left\{ \begin{array}{l} \text{closed subschemes of } X \end{array} \right.$$

$$\downarrow$$

$$X$$

Cat which is a set.

descent means: $X \xrightarrow{f} Y$ cov

$Z \in CL(Y) \longleftrightarrow Z' \in CL(X)$ s.t.

$$\pi_1^* Z' = \pi_2^* Z'$$

$$Y \times_X Y \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Y$$

why?

$$Z \in CL(X)$$

$$\updownarrow$$

$$I_Z \in \text{Ideals}(\mathcal{O}_X) \text{ qcoh.}$$

$(I_Z \hookrightarrow \mathcal{O}_X)$ follows from q-coh. descent.

$$I \triangleleft \mathcal{O}_Y \longleftrightarrow I' \triangleleft \mathcal{O}_X \text{ \& } \pi_1^* I' = \pi_2^* I'$$

(flat!)

Descent for affine morphisms

$$\begin{array}{c} AFF \\ \downarrow \\ S_{fppf} \end{array}$$

$AFF(X)$ $(f : X' \to X)$ affine.

$$\updownarrow$$

qcoh sheaf of $\mathcal{O}_X$-algebras.

D

Descent for Polarized morphisms

$$\begin{array}{c} Pol \\ \downarrow \\ S \end{array} \qquad Pol(Y) \qquad (X \xrightarrow{f} Y,\ \mathcal{L} \text{ line bundle on } X \text{ f-ample})$$

then we have fppf descent.

Pf sketch: only interesting part is how to glue $\mathcal{L}$ to get ample. i.e. given descent data:

$Y' \xrightarrow{g} Y$ fppf cover & have $Y'' = Y' \times_Y Y'$

$$\begin{array}{c} Y'' \\ \downarrow \pi_1, \pi_2 \\ Y' \end{array}$$

$(f' : X' \to Y',\ \mathcal{L}'/X'$ rel. ample$)$ & isom's (α, β, γ) $X' \dashrightarrow Y'$

$$\begin{tikzcd} \pi_1^* X' \arrow[r, "\pi_1 f'"] \arrow[d, "\alpha"'] & \pi_1^* Y' \arrow[d, "\beta"] \\ \pi_2^* X' \arrow[r, "\pi_2^* f'"'] & \pi_2^* Y' \end{tikzcd}$$

commutes &

$$\begin{array}{c} \pi_1^* \mathcal{L}' \\ \downarrow \gamma \\ \pi_2^* \mathcal{L}' \end{array}$$

s.t.

$$\pi_{13}^* \alpha = \pi_{23}^* \alpha \quad \pi_{12}^* \alpha$$

β

γ

then $\exists$

$(X \xrightarrow{f} Y,\ \mathcal{L}/X$ rel. amp$)$

amples go local on Y. $\rightsquigarrow$ glue proj embeddings. $\square$.