Lecture 14: More descent (part 5)

Thursday, October 2, 2014 11:04 AM

Example Moduli spaces of curves

(absolute not relative)

fix g = genus $\neq 1$

$\mathcal{M}_g$ cat w/ objects $(C \xrightarrow{f} S)$ f flat proper, w/ $f^{-1}(s)$ = smooth genus g curve over s = Spec k geom. irreducible.

$$\begin{array}{ccc} \mathcal{M}_g & & (C \to S) \\ \downarrow & & \downarrow \\ Sch & & S \end{array}$$

morphisms:

$$\begin{tikzcd} C' \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \longleftrightarrow (C' \to S') \to (C \to S)$$

pullback

Observe: morphisms of fibred cats $\mathcal{M}_g \longrightarrow Pol$

$g=0$ $(C \to S) \to (C \to S, {\Omega^1_{C/S}}^{\otimes -1})$

$g \geq 2$ $(C \to S) \to (C \to S, \Omega^1_{C/S})$

Using this, can see that $\mathcal{M}_g$ has effective descent for fppf topology

$$\begin{array}{ccc} \mathcal{M}_g(S) & \longrightarrow & Pol(S) \\ \downarrow & & \downarrow \wr \\ \mathcal{M}_g(\{S' \to S\}) & \longrightarrow & Pol(\{S' \to S\}) \end{array}$$

$\square$

Torsors $\mathcal{C}$ a site, μ = sheaf of groups on $\mathcal{C}$

A μ-torsor is a sheaf P w/ left μ-action

$(\mu \times P \to P)$

s.t. 1. for each $X \in \mathcal{C}$, $\exists$ cover $\{X_i \to X\}$ s.t.

$P(X_i) \neq \emptyset$

2. $\mu \times P \to P \times P$
$(g, p) \mapsto (gp, p)$ is an isom. of sheaves

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ex: $\mathcal{C}$ fppf μ = group scheme, $\begin{matrix} \mu \\ \downarrow \text{fppf} \\ S \end{matrix} \overset{\text{using } 2.}{\rightsquigarrow} \begin{matrix} P \\ \downarrow \text{fppf} \\ S \end{matrix}$

P representable

2. says $\{P \to S\}$ $P(P) \neq \emptyset$ $(2 \Rightarrow 1)$

<u>μ-tors</u> objects: P s.t. 1), 2)

morphisms. $P_1 \xrightarrow{f} P_2$ s.t.

$$\begin{array}{ccc} \mu \times P_1 & \longrightarrow & P_1 \\ \downarrow \scriptstyle{id \times f} & & \downarrow \scriptstyle{f} \\ \mu \times P_2 & \longrightarrow & P_2 \end{array}$$

commutes.

<u>Def:</u> A principal G-bundle (G gp scheme / X) is a pair $(P \xrightarrow{\pi} X, \sigma: G \times_X P \to P$ (of X-schemes))

s.t. σ an action:

i)
$$\begin{tikzcd} G\times G\times P \arrow[r, "m, id"] \arrow[d, "id\times\sigma"'] & G\times P \arrow[d, "\sigma"] \\ G\times P \arrow[r, "\sigma"'] & P \end{tikzcd}$$
commutes

ii) id acts trivially..

iii) $G\times_X P \longrightarrow P\times_X P$ isom.
$(g,p) \longrightarrow (gp, p)$

Yoneda: (princ G-bundle) $\rightsquigarrow$ (μ-torsor)
$\mu = h_G$

Prop (from what we know so far) if G/X affine $\Rightarrow$ equiv.

Example Consider: $\mu = \mathbb{G}_m$ (fppf)
$\mathbb{L}$ = cat of line bundles on X (thought of as affine morphisms)

Will define an equiv.
$$\mathbb{L} \longrightarrow \mathbb{G}_m\text{-torsors}/X.$$

$$\mathcal{L} \longmapsto \text{triv}(\mathcal{L}) \left\{ \begin{array}{l} U \to X \\ \text{triv}(\mathcal{L})(U) = \{\text{isoms } \mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U\} \end{array} \right.$$

$\mathcal{L}$ loc. trivial $\rightsquigarrow$ $\text{triv}(\mathcal{L})$ locally nonempty

$\mathbb{G}_m$ act on any sheaf of modules
$\circlearrowright \mathcal{O}$

$$\mathbb{G}_m \times \mathrm{triv}(\mathcal{L}) \to \mathrm{triv}(\mathcal{L}) \times \mathrm{triv}(\mathcal{L})$$
$$g, \sigma \longmapsto g\sigma, \sigma$$

locally is $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$
$$(a, b) \to (ab, b)$$

locally $\mathcal{L} \simeq \mathcal{O}_X$ $\quad$ $\mathrm{triv}(\mathcal{L})$ locally $\mathrm{triv}(\mathcal{O}_X)$

$$\boxed{\mathbb{G}_m(S) = \mathcal{O}_S^*} \qquad \mathcal{O}_X \to \mathcal{O}_X$$

for other direction, given a $\mathbb{G}_m$ torsor P get a line bundle via:

choose $\{X_i \to X\}$ s.t. $P(X_i) \neq \emptyset \Rightarrow (P(X_i \times_X X_j) \neq \emptyset)$
pick $x_i \in P(X_i)$

2) says that there is a $\varphi_{ij} \in \mathbb{G}_m(X_i \times_X X_j)$ s.t.

$\varphi_{ij}(x_i) = x_j$ $\qquad$ x_i, x_j give two sections in $P(X_i \times_X X_j)$

$\pi_1^* x_i, \pi_2^* x_j$
$\quad \| \qquad \|$
$\quad x_i \qquad x_j$

$$\mathbb{G}_m \times P \xrightarrow{\sim} P \times P$$
$\varphi_{ij} \qquad x_i \qquad x_j \quad x_i$

φ_{ij}'s satisfy cocycle, define isom's $\mathcal{O}_{X_i}|_{X_i \times X_j} \to \mathcal{O}_{X_j}|_{X_i \times X_j}$

gluing data for line bundle.

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eg: If we had used GL_n, would have been talking about vector bundles rk n

($PGL_n \rightsquigarrow$ proj. bundles)

μ_n - torsors $\qquad X$ scheme, $n \in \mathcal{O}_X(X)^*$

μ_n defined as kernel $\quad \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$

Σ_n cat of pairs (L, σ), $\sigma : L^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X$

L a line bundle

(étale top)

morphisms $\quad (L', \sigma') \longrightarrow (L, \sigma)$ $\qquad f : L' \to L$

$$\begin{tikzcd} L'^{\otimes n} \arrow[d, "f^{\otimes n}"'] \arrow[dr, "\sigma'"] & \\ L^{\otimes n} \arrow[r, "\sigma"'] & \mathcal{O}_X \end{tikzcd}$$