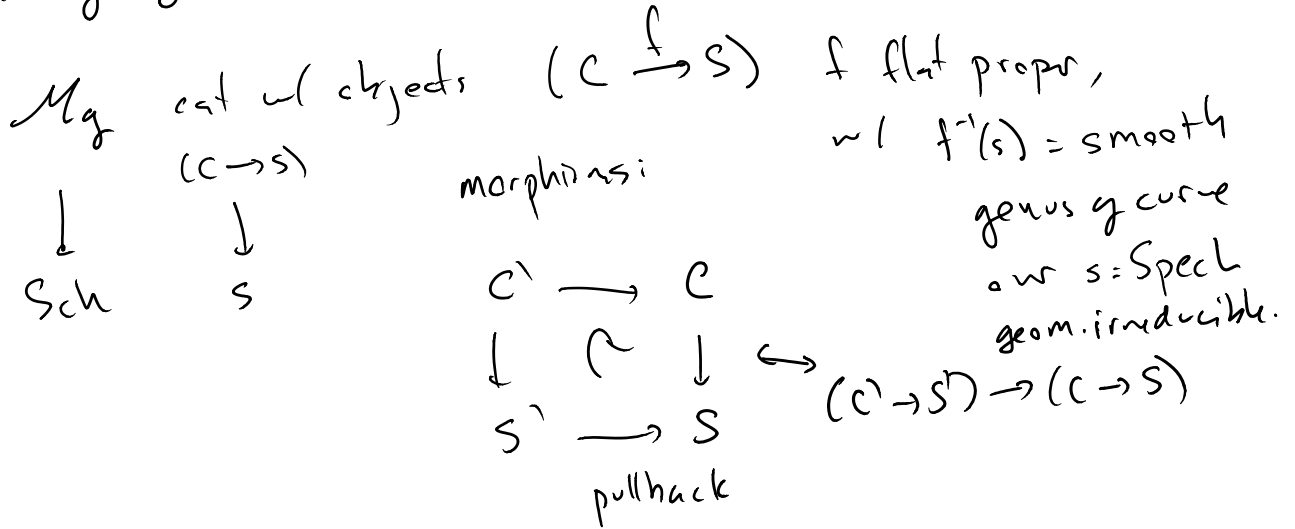


Example Moduli spaces of curves

fix $g = \text{genus} \neq 1$ (absolute not relative)

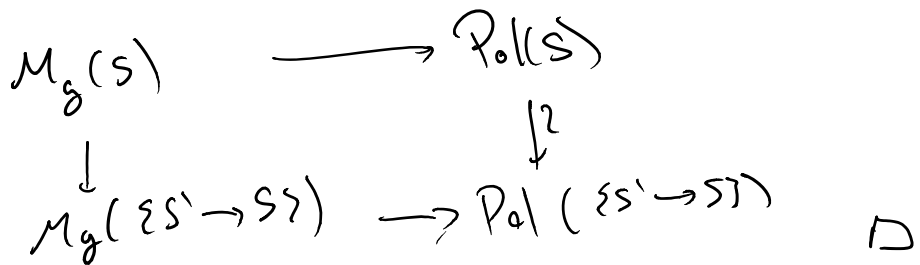


Observe: morphisms of fibered cats $M_g \rightarrow \mathcal{P}ol$

$$g=0 \quad (C \rightarrow S) \rightarrow (C \rightarrow S, \Omega_{C/S}^{g-1})$$

$$g \geq 2 \quad (C \rightarrow S) \rightarrow (C \rightarrow S, \Omega_{C/S}^1)$$

Using this, can see that M_g has effective descent for fppf topology



Torsors \mathcal{C} a site, $\mu = \text{sheaf of groups on } \mathcal{C}$

A μ -torsor is a sheaf \mathcal{P} w/ μ -action
left
($\mu \times \mathcal{P} \rightarrow \mathcal{P}$)

s.t. 1. for each $X \in \mathcal{C}$, \exists cover $\{X_i \rightarrow X\}$ s.t.
 $\mathcal{P}(X_i) \neq \emptyset$

2. $\mu \times \mathcal{P} \rightarrow \mathcal{P} \times \mathcal{P}$
 $(g, p) \mapsto (gp, p)$ is an isom. of
schemes

ex: \mathcal{C} fppf $\mu = \text{group scheme}$, \mathcal{P} representable
2. says $\{\emptyset \rightarrow S\}$ $\mathcal{P}(\emptyset) \neq \emptyset$ (2 \Rightarrow 1)

μ -tors

objects: \mathcal{P} s.t. 1), 2)

morphisms: $\mathcal{P}_1 \xrightarrow{f} \mathcal{P}_2$ s.t.

$$\begin{array}{ccc} \mu \times \mathcal{P}_1 & \longrightarrow & \mathcal{P}_1 \\ \downarrow \text{id} \times f & & \downarrow f \\ \mu \times \mathcal{P}_2 & \longrightarrow & \mathcal{P}_2 \end{array} \quad \text{commutes.}$$

Def: A principal G -bundle (G gp scheme / X) is a
pair $(\mathcal{P} \xrightarrow{\pi} X, \sigma: G \times_X \mathcal{P} \rightarrow \mathcal{P} \text{ (of } X\text{-schemes)})$

s.t. σ an action

$$\begin{array}{ccc}
 G \times G \times P & \xrightarrow{m, id} & G \times P \\
 \sigma \times \sigma \downarrow & & \downarrow \sigma \\
 G \times P & \xrightarrow{\sigma} & P
 \end{array}$$

Commutative
 (i) id -acts trivially...

(ii) $G \times_x P \rightarrow P \times_x P$ isom.
 $(g, p) \rightarrow (gp, p)$

Yoneda: $(\text{prin } G\text{-bundle}) \rightsquigarrow (\mu\text{-torsor})$
 $\mu = h_G$

Prop (from what we know so far) if G/X affine \Rightarrow equiv.

Example Consider: $\mu = G_m$ (fppt)
 $\mathcal{L} = \text{cat of line bundles on } X$ (thought of as affine morphisms)

Will define an equiv.

$$\mathcal{L} \longrightarrow G_m\text{-torsors}/X.$$

$$\mathcal{L} \xrightarrow{\text{triv}(\mathcal{L})} \left\{ \begin{array}{l} u \rightarrow X \\ \text{triv}(\mathcal{L})(u) = \{ \text{isom } \mathcal{L}|_u \xrightarrow{\sim} \mathcal{O}_u \} \end{array} \right.$$

\mathcal{L} loc. trivial \rightsquigarrow $\text{triv}(\mathcal{L})$ locally

nonempty

G_m act on any shift of module

$\circ \mathcal{O}$

$$G_m \times \text{triv}(Z) \rightarrow \text{triv}(Z) \times \text{triv}(Z)$$

$$g, \sigma \longmapsto g\sigma, \sigma$$

locally is $G_m \times G_m \rightarrow G_m \times G_m$

$$(a, b) \rightarrow (ab, b)$$

locally $Z \cong \mathcal{O}_x$ $\text{triv}(Z)$ locally $\text{triv}(\mathcal{O}_x)$

$$\boxed{G_m(S) = \mathcal{O}_S^*}$$

$$\mathcal{O}_x \rightarrow \mathcal{O}_x$$

for other direction, given a G_m torsor \mathcal{P} get a line bundle

via:

choose $\{X_i \rightarrow X\}$ s.t. $\mathcal{P}(X_i) \neq \emptyset \Rightarrow (\mathcal{P}(X_i \times X_j) \neq \emptyset)$

pick $x_i \in \mathcal{P}(X_i)$

2) says that there is a $\varphi_{ij} \in G_m(X_i \times X_j)$ s.t.

$$\varphi_{ij}(x_i) = x_j$$

x_i, x_j give two sections in $\mathcal{P}(X_i \times X_j)$

$$G_m \times \mathcal{P} \xrightarrow{\sim} \mathcal{P} \times \mathcal{P}$$

φ_{ij} x_i x_j x_i x_j

$$\begin{matrix} \pi_1^* x_i, \pi_2^* x_j \\ \parallel & \parallel \\ x_i & x_j \end{matrix}$$

\mathcal{O}_{x_i} 's cocycle, define isom's $\mathcal{O}_{x_i}|_{X_i \times X_j} \rightarrow \mathcal{O}_{x_j}|_{X_i \times X_j}$

gluing data for line bundle.

ex: If we had used GL_n , would have been talking about
vector bundles
of rank n

($PGL_n \rightarrow$ proj. bundles)

M_n -torsors X scheme, $\mathcal{O}_X(X)^*$
 M_n defined as kernel $GL_n \xrightarrow{u} GL_n$

Σ_n cat of pairs (L, σ) , $\sigma: L^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X$
 L a line bundle

(étale top)

