Lecture 15: end of descent, beginning of algebraic spaces

Tuesday, October 7, 2014 11:12 AM

μ_n-torsors (X étale site (big))

X scheme, $n \in \mathcal{O}_X(X)^*$, μ_n = sheaf of n'th roots of unity

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{\cdot n} \mathbb{G}_m \to 1$$

Σ_n cat. whose objects are pairs $(\mathcal{L}, \sigma)$

$\mathcal{L}/X$ invertible sheaf, $\sigma : \mathcal{L}^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X$

morphisms are $(\mathcal{L}', \sigma') \xrightarrow{f} (\mathcal{L}, \sigma)$

$f : \mathcal{L}' \to \mathcal{L}$ s.t.

$$\begin{tikzcd} \mathcal{L}'^{\otimes n} \arrow[r, "\sigma'"] \arrow[d, "f^{\otimes n}"'] & \mathcal{O}_X \\ \mathcal{L}^{\otimes n} \arrow[ur, "\sigma"'] & \end{tikzcd}$$ commutes.

Given $(\mathcal{L}, \sigma)$ as above, get a μ_n-torsor $P_{(\mathcal{L},\sigma)}$

via. $P_{(\mathcal{L},\sigma)}(U) = \left\{ \lambda : \mathcal{O}_U \xrightarrow{\sim} \mathcal{L}|_U \;\middle|\; \begin{tikzcd} \mathcal{O}_U \arrow[r, "id"] \arrow[d, "\lambda^{\otimes n}"'] & \mathcal{O}_U \\ \mathcal{L}_U^{\otimes n} \arrow[ur, "\sigma"'] \arrow[r, phantom, "C"] & \end{tikzcd} \right\}$

$$\begin{tikzcd} U \arrow[d] \\ X \end{tikzcd}$$

changing lambda by mult. w/ an n^{th} root of unity $\lambda \to \zeta \cdot \lambda$ gives an action of μ_n on $P_{(\mathcal{L},\sigma)}(U)$.

$\mu_n \quad \cdot \quad {}_{(\mathcal{L},\sigma)} \cdots$

note either $\mathrm{Isom}(\mathcal{O}_U, \mathcal{L}|_U) = \begin{cases} \emptyset \\ \mathrm{Isom}(\mathcal{O}_U, \mathcal{O}_U) = \mathcal{O}_U^* = \mathbb{G}_m(U) \end{cases}$

has an action by $\mathbb{G}_m$
simply transitive (when nonempty)

the only elmts in $\mathbb{G}_m$ which make diagram commute differ by elmts in μ_n

Haven't shown why $P_{(\mathcal{L},\sigma)}$ has sections locally:

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "f^{\otimes n}"] \arrow[rr, bend right, "id"'] & \mathcal{L}^{\otimes n} \arrow[r, "\sigma"] & \mathcal{O}_X \end{tikzcd}$$

$$\mathcal{O}_X \xrightarrow{f} \mathcal{L}$$

locally (Zariski) $\qquad \mathcal{L} = \mathcal{O}_X$

$$\mathcal{O}_X \xrightarrow{\cdot a^n} \mathcal{O}_X \xrightarrow{\sigma = s \in \mathcal{O}_X^*} \mathcal{O}_X$$

$$\mathcal{O}_X \xrightarrow{f = a \in \mathcal{O}_X^*} \mathcal{O}_X \qquad s \cdot a^n = 1 \in \mathcal{O}_X$$

$P_{(\mathcal{L},\sigma)}(\text{little}) = \{ a \in \mathcal{O}_X^* \mid a^n = s \}$

define $Y \to X$ via taking n^{th} root of s, get étale cover.

ex: $X = \operatorname{Spec} k \qquad \mu_n$-torsors on $X \qquad \sigma = s$

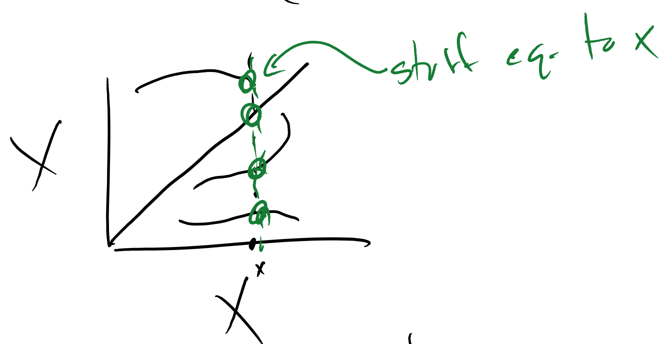
$\qquad {}^{*} \qquad \cdots$

ex: $X = \operatorname{Spec} k$ $\quad$ μ_n-torsors on X $\longleftrightarrow$ $k^*/(k^*)^n \longleftrightarrow$ "s"

Algebraic Spaces

An algebraic space is, roughly something of the form

X/R $\quad$ X scheme, R equivalence relation (étale)



X/R we really mean $U \rightsquigarrow h_X(U) / h_R(U)$ ← set ← eq. rel

get presheaf, sheafify (étale)

$\rightsquigarrow X/R$

Nice quotients by finite gps

$G \circlearrowright X$ $\quad$ if X is q-proj $\Rightarrow$ can find an affine cover

$G \circlearrowright X$ if X is q-proj, then

$$\bigcup X_i = X \qquad \operatorname{Spec} R_i = X_i$$

s.t. $G \circlearrowright X_i$ $\qquad$ G acts on R_i

$$X_i/G = \operatorname{Spec} R_i^G$$

glue these, get X/G

X not q-proj $\rightsquigarrow$ may (usually) happen that X/G not a scheme.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"] & Y \\ X \arrow[ur] & \end{tikzcd}$$

Shift to go "coordinate free" – use diagonal

general <u>comment</u>:

fiber products of schemes agree w/ fiber products of sheaves

" ——— sheaves " ——— presheaves.

In particular: if F, T, T' sheaves

$$\begin{tikzcd} & T \times T' \arrow[d] \\ F \arrow[r, "\Delta"] & F \times F \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[d, "f"] \\ T' \arrow[r, "g"] & F \end{tikzcd}$$

fiber products agree.

ex of this: $F = X/R$

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \times X \arrow[d] \\ F \arrow[r, "\Delta"] & F \times F \end{tikzcd} \qquad \begin{tikzcd} \boxed{R} \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & F \end{tikzcd}$$

Punchline – if F is of above form, give $X \to F$ recover R as $F \times_{F \times F} (X \times X) = R$

Language – stability, closedness, locality on domain

stability ← covg on range; closedness ← pullbacks; locality on domain ← covg on domain

C a site: (assume that all representable presheaves are sheaves)

Def A class of objects $\mathcal{S} \subset Ob(C)$ is stable if $\forall$ covs $\{U_i \to U\}$, $U \in \mathcal{S} \Leftrightarrow$ each $U_i \in \mathcal{S}$.

Example C - Zariski loc. Noeth, reduced, normal, regular.

Def $\mathcal{P} \subset C$ subcat is closed if

1) $\mathcal{P}$ contains iso's.

2) closed under pullbacks $U \xrightarrow{\text{(cartsian)}} V$?

1) [illegible]

2) closed under pullbacks

$f \in \mathcal{D} \Rightarrow f' \in \mathcal{D}$

(cartesian)

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f'"'] & V \arrow[d, "f"] \\ X \arrow[r] & Y \end{tikzcd}$$

<u>Def</u> $\mathcal{D} \subset \mathcal{C}$ subcat is stable if closed & if for each $f: X \to Y$ in $\mathcal{C}$, cov $\{Y_i \to Y\}$, then $f \in \mathcal{D} \Leftrightarrow f_i: X \times_Y Y_i \to Y_i$ in $\mathcal{D}$.

<u>Def</u> $\mathcal{D} \subset \mathcal{C}$ subcat is local on domain, if $\mathcal{D}$ is stable & if $f: X \to Y$, $\{X_i \to X\}$ cov then $f \in \mathcal{D} \Leftrightarrow X_i \to X \to Y$ in $\mathcal{D}$ all i.

<u>ex:</u> S scheme $\mathcal{C} = (Sch/S)_{\acute{e}t}$ then

<u>Stable:</u> proper, separated, surjective, q-compact.

<u>Local on Domain:</u> loc. finite type, loc. f. pres., flat, étale, univ. open, loc. q. finite, smooth.