Lecture 16: algebraic spaces. definitions constructions and examples

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<u>Def.</u> S scheme, $F, G \in$ Sheaves on $(Sch/S)_{ET}$
S_{Et}

$f: F \to G$ morphism is representable if $\forall\, T \in Sch/S$
& $T \to G$ morphism then $F \times_G T$ is a scheme.
(h_T)

$$F \times_G T\,(U) = \{(\sigma, \varphi) \mid \sigma \in F(U),\ \varphi: U \to T,\ f(\sigma) = g_U\}$$

$T \to G \longmapsto G(T) \ni g$

scheme

$$\begin{tikzcd} F \times_G T \arrow[r] \arrow[d] & F \arrow[d] \\ T \arrow[r] & G \end{tikzcd}$$

If P is a stable property of morphisms (preserved by pullbacks)
then if f is representable, we say f has prop P if
every pullback as above (by a scheme) does.

<u>Lemma</u> S a scheme F a sheaf in S_{et}, suppose
$\Delta: F \to F \times F$ is representable. Let $f: T \to F$ be
any morphism, with T an S-scheme. Then f is
representable.

Pf: WTS fiber product

$$\begin{array}{ccc} & & T \\ & & \downarrow f \\ T' & \xrightarrow{g} & F \end{array}$$

is rep. all $T' \to F$, S-scheme

fiber prod =

$$\begin{array}{ccc} & & T \times_S T' \\ & & \downarrow f \times g \\ F & \xrightarrow{\Delta} & F \times F \end{array}$$

which is rep, since Δ is rep. $\square$

<u>Def</u> Let S be a scheme. An alg. space over S is a functor $X : (Sch/S)^{op} \longrightarrow Set$ s.t.

i) X is a sheaf in the big étale top

ii) $\Delta : X \longrightarrow X \times X$ is representable (by schemes)

iii) $\exists$ an S-scheme $U \to S$ & a surjective étale morphism $U \longrightarrow X$.

<u>Recall:</u> (S fixed base scheme)

an equiv. rel on X/S is a monomorphism of (sheaf) schemes $R \hookrightarrow X \times_S X$

s.t. i) $\forall$ T/S, $R(T) \hookrightarrow X(T) \times X(T)$

is an eq. rel on $X(T)$

it's called étale if

ii) the maps

$$\begin{tikzcd} & & X \\ R \arrow[r] \arrow[urr, bend left, "s"] \arrow[drr, bend right, "t"'] & X \times_S X \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] & \\ & & X \end{tikzcd}$$

s, t are étale.

Def Given an étale eq. rel R on X/S

we can consider the presheaf

$$T \longrightarrow X(T)/R(T)$$

we write X/R for the sheafification.

Prop i) X/R is an algebraic space.

ii) If Y/S an alg space, X a scheme$/S$, $X \to Y$ is étale surjective

then $R \equiv X \times_Y X$ is an étale eq. rel.

and the natural map $X/R \to Y$ an isom.

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ex: X a scheme, G a finite group acting on X

$G \times X \xrightarrow{\rho} X \times X$ image $= R$

$(g,x) \longmapsto (gx, x)$

we say that the action is __free__ if ρ is a monomorphism.

i.e. $R \simeq G \times X$ $\qquad$ $\begin{array}{c} G \times X \\ \downarrow s,t \\ X \end{array}$ étale.

free $\Rightarrow$ R étale eq. rel.

$$X/G \equiv X/R \quad \text{is an alg space.}$$

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ex. ex: $X = \mathbb{A}^1_k$ $\quad$ $G = \mathbb{Z}$ acting by translation

char $k = 0$

$\mathbb{Z} \times X \longrightarrow X \times X$

free, étale eq rel $\quad$ $\mathbb{A}^1_k / \mathbb{Z}$

not a scheme.

f. field would be in $k(x)^{\mathbb{Z}}$

$x \mapsto x+1$

f. field is k. $\rightsquigarrow$ scheme quot. is a pt, Spec k.

Proof of prop:

First part: Given X, R (S-scheme & étale eq. rel)

$Y = X/R$ want to show Y is an alg. space.

main thing to prove is $Y \xrightarrow{\Delta} Y \times Y$ representable.

Why does rep. of Δ give everything else?

Assume Δ representable. Need to show:

$\exists\ U \to Y$ étale surjective. U/S scheme.

Consider $X \to X/R = Y$ is surjective. want to show it's étale.

i.e. consider $T \to Y$ (representable)

want

$$\begin{tikzcd} X \times_Y T \arrow[r, "\text{étale}"] \arrow[d] & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

to check can replace T w/ étale cover.

i.e. since $X \to Y$ surj as étale sheaves

$$\begin{tikzcd} X \arrow[r] & Y \\ X' \arrow[u, dashed] & T \arrow[u] \end{tikzcd}$$

shrinking T (étale) can assume $T \to Y$ factors through X.

$$T \to X \to Y$$

$$\begin{tikzcd} T \times_Y X \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & X \arrow[d] \end{tikzcd}$$

$$
\begin{tikzcd}
1 \times_y \wedge \arrow[r] \arrow[d, "\text{ét}?"'] & K \arrow[d, "\text{ét.}"'] & \wedge \arrow[d] \\
T \arrow[r] & X \arrow[r] & y
\end{tikzcd}
$$