Lecture 18: Properties of Algebraic Spaces

Thursday, October 16, 2014 11:00 AM

<u>Today</u>
- Finish spaces as quotients
- Fibor products of spaces
- Alg. spaces are fppf sheaves

S = base scheme

<u>Prop</u> i) Let X be a scheme, $R \subset X \times X$ étale eq. rel. $Y = X/R$ then Y is an alg space

ii) If Y is an alg. space, $X \to Y$ ét. surj map w/ X a scheme, then $R = X \times_Y X$ is an ét. eq. rel & $X/R \cong Y$.

<u>Last time</u>: i) showed that Δ_Y representable, time before last, showed this $\Rightarrow$ i).

ii) $R = X \times_Y X$ then

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \times_S X \arrow[d] \\ Y \arrow[r, "\Delta"] & Y \times Y \end{tikzcd}$$

Y alg. space Δ rep. $\Rightarrow$ R scheme

want $R \to X \times X \to X$ étale

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\text{ét}"] & X \arrow[d, "\text{ét}"] \\ X & Y \end{tikzcd}$$

therefore $R \to X \times X$ ét. eq. rel

i) $\Rightarrow X/R$ alg. space

$$\begin{array}{ccc} \downarrow \text{ét} & \Leftarrow & \downarrow \text{ét} \\ X & \longrightarrow & Y \end{array}$$

i) $\Rightarrow X/R$ ...

$X/R \longrightarrow Y$ immediate by looking at it sheaf-wise.

$\square$

<u>Def</u> If P is a prop. of schemes which is stable in étale top (true $\Leftrightarrow$ true for cover) then we say an alg. space X has prop. $P \Leftrightarrow \exists\, U \to X$ ét. surj., U a scheme

U has P.

<u>ex:</u> loc. noeth, reduced, regular, n-dim'l, normal.

<u>Def</u> If P is a prop. of morphisms stable in ét. top on base, representable by schemes, then if $f : X \longrightarrow Y$ is a morphism of alg. spaces, we say f has P if $\exists\, U \longrightarrow Y$ ét surj s.t. w/ U a scheme

$X \times_Y U \longrightarrow U$ has P.

<u>ex:</u> closed embeddings, loc. closed embedding, open embeddings, embeddings, proper, dominant, q-compact.

<u>Def</u> $f : X \longrightarrow Y$ morph. of alg. spaces is q-separated (loc. sep., sep) if $\Delta_{X/Y}$ is q-comp. (loc. closed embedding, closed embedding)

if $\Delta_{X/Y}$ is q.comp ...

<u>Ex</u> $X = \mathbb{A}^1_k$ action of $\mathbb{Z}$ on X by $x \mapsto x+n$

char $k = 0$

$Y = X/\mathbb{Z}$ $\quad$ $R = \mathbb{A}^1_k \sqcup (\mathbb{A}^1_k)_1 \sqcup (\mathbb{A}^1_k)_{-1} \sqcup \cdots$

$$\begin{array}{ccc} R & & (x)_n \\ \downarrow & & \downarrow \\ \mathbb{A}^1_k \times \mathbb{A}^1_k & & (x, x+n) \end{array}$$

$\uparrow$ exists as an alg. space.

∞ly many components

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[d] \\ Y \arrow[r] & Y \times_S Y \end{tikzcd}$$

$\Rightarrow$ Y not quasi-separated.

Can also talk about prop. of morphisms stable & local on domain for alg. spaces. (morph. need not be rep).

$$\begin{tikzcd} X \arrow[d, "\text{étale?}"] \\ Y \end{tikzcd}
\qquad
\begin{tikzcd} & & V \arrow[dl, "\text{ét cover}"'] \arrow[ddl, dashed, "\text{ét?}"] \arrow[ddl, bend left] \\ X \arrow[d] & X \times_Y U \arrow[l] \arrow[d, "\text{ét?}"] & \\ Y & U \arrow[l, "\text{ét cover}"] & \end{tikzcd}$$

$\uparrow$ schemes

$\underline{ex}$: étale, flat, smooth, surj.

Fiber products

$\underline{\text{Prop}}$: the fiber products of algebraic spaces over alg. spaces are alg. spaces.

i.e. given algebraic spaces
$$\begin{tikzcd} & X_1 \arrow[d] \\ X_2 \arrow[r] & X_3 \end{tikzcd}$$
, then $X_1 \times_{X_3} X_2$ is an alg. space.

$$\begin{tikzcd}
X_1 \times_{X_3} X_2 \arrow[dr, hook, "\text{monomorphism of sheaves}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_1 \times_S X_2 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
& X_2 \arrow[r] & X_3 \arrow[dr] \\
& & & S
\end{tikzcd}$$

begin w/ case $X_3 = S$

check rep. of $\Delta_{X_1 \times X_2}$

$$\begin{tikzcd}
& U \arrow[d] \\
X_1 \times X_2 \arrow[r, "\Delta"] \arrow[d, equal] & (X_1 \times X_2) \times (X_1 \times X_2) \arrow[d, equal] \\
{} & {}
\end{tikzcd}$$

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "\Delta_{X_1} \times \Delta_{X_2}"] & (X_1 \times X_1) \times (X_2 \times X_2) \end{tikzcd}$$

rep. since $\Delta_{X_1}, \Delta_{X_2}$ rep.

so fibr prod is a scheme,

further, if $U_i \to X_i$ $i = 1, 2$ étale surj. U_i scheme,

then $U_1 \times U_2 \to X_1 \times X_2$ ét. surj.

$\Rightarrow X_1 \times X_2$ an alg. space.

For general X_3,

$$\begin{tikzcd} \boxed{\text{Thing}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d] \\ X_1 \times_{X_3} X_2 \arrow[r] & (X_1 \times_{X_3} X_2) \times (X_1 \times_{X_3} X_2) \\ X_1 \times X_2 \arrow[r] \arrow[u, hook] & (X_1 \times X_2) \times (X_1 \times X_2) \arrow[u, hook] \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \arrow[d] \\ X_1 \times X_2 \arrow[r] & X_1 \times X_2 \times X_1 \times X_2 \end{tikzcd}$$

(from before) no X_3's.

$$\begin{tikzcd} \boxed{\mathrm{Thy}} \arrow[r] \arrow[d] & R \arrow[d] \\ X_3 \arrow[r, "\Delta"] & X_3 \times X_3 \end{tikzcd}$$

Δ_{X_3} rep $\Rightarrow$ $\boxed{\mathrm{Thy}}$ is a scheme

punchline for $\exists$ $U \longrightarrow X_1 \times_{X_3} X_2$ etale surj, U_3 scheme.

$$\begin{tikzcd} \text{pull it back} \arrow[r] \arrow[d] & \tilde{U} \arrow[d] \\ X_1 \times_{X_3} X_2 \arrow[r, hook] & X_1 \times X_2 \end{tikzcd}$$

<u>Next goal:</u> Alg. spaces are fppf sheaves.

<u>Thm</u> S scheme, X/S alg. space w/ q. comp. diagonal. then X is an fppf sheaf.

<u>Proof:</u> let $\overline{X}$ be the fppf sheafification of X

$X \longrightarrow \overline{X}$. want to show its a bijection on sections.

injectivity, suppose $x_1, x_2 \in X(U)$

sub of U s.t. $x_1 = x_2$

$$\begin{tikzcd} U \arrow[d, "x_1, x_2"] \\ {} \end{tikzcd}$$

$x_1 = x_2$ means fiber prod is U.

$$X \xrightarrow{\Delta} X \times X$$

ét show, same fiber product as in

$$\begin{tikzcd} & U \arrow[d] \\ \bar{X} \arrow[r] & \bar{X} \times \bar{X} \end{tikzcd}$$

to make latter, make a presheaf fiber product then sheafify.

$$\overline{U \times_{(X \times X)} X} = \bar{U} \times_{(\bar{X} \times \bar{X})} \bar{X} = U \times_{(\bar{X} \times \bar{X})} \bar{X}$$

$\parallel$

fppf sheafification of

$U \times_{(X \times X)} X$ which is a scheme

so $\overline{U \times_{(X \times X)} X} = U \times_{(X \times X)} X$

surjectivity

$$\begin{tikzcd} \bigcirc \arrow[r] \arrow[d] & X_0 \times_{\bar{X}} U \arrow[r] \arrow[d, "ét?"'] & X_0 \arrow[d, "ét"] \\ & & X \arrow[d] \\ \tilde{U} \arrow[r, "fppf"'] & U \arrow[r] & \bar{X} \end{tikzcd}$$

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