Lecture 19: Sprint through algebraic spaces, part 1

Tuesday, October 21, 2014 11:01 AM

__Def__ $X_1 \overset{t}{\underset{s}{\rightrightarrows}} X_0$ groupoid in schemes

$(X_1 \to X_0 \times X_0)$

then $f: X_0 \to T$ is __invariant__ if the compositions

$$\begin{tikzcd} X_1 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] \arrow[rr, bend left] \arrow[rr, bend right] & X_0 \arrow[r] & T \end{tikzcd}$$

are equal

$f: X_0 \to T$ universally invariant if it's invariant and for all $g: X_0 \to T'$ invariant, we have

$(S, T, X_0, X_1$ all schemes$)$

$$\begin{tikzcd} X_0 \arrow[r, "g"] \arrow[d, "f"'] & T' \\ T \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

__Thm__ 6.2.2 assume that $X_1 \overset{t}{\underset{s}{\rightrightarrows}} X_0$ groupoid w/ s, t, finite flat and for every $x \in X_0$ $s(t^{-1}(x))$ is contained in an affine of X_0.

then $\exists$ a universal invariant morphism $X_0 \to T$.

In fact, it will be universal for loc. ringed spaces.

__Consequence (?)__

__Thm 6.4.1__ X/S alg. space, q. sep. Then $\exists\, V$ scheme, and a dense open embedding $V \hookrightarrow X$.

and a dense open embedding $U \hookrightarrow X$.

Moral: Spaces are birat'l to schemes.

next time { Can talk about function fields (irred), etc.
In fact, given an alg-space, assoc. a top space.

Quasi-Coherent sheaves & Coherent sheaves

Def X alg space/S. The small étale site of X has underlying category the set of étale morphisms of alg. spaces $Y \to X$ & covs = families $\{Y_i \to Y\}$ s.t. $\coprod Y_i \to Y$ is surjective.

$Et(X)$ site. $X_{ét}$ = topos.

Variations: • Big étale site (all $Y \to X$ alg. space morphisms for underlying cat

same covs.

• $Et'(X) \subset Et(X)$ subcat of morphisms $Y \to X$
Y scheme.

$X_{ét'} \leftarrow X_{ét}$ is an equiv. of topoi!

to define sheaf on $X_{ét}$ suffices to define one on $X_{ét'}$

$$R \rightrightarrows U \to X$$
$$\parallel$$
$$U \times_X U$$

$$\mathcal{F}(X) = Eq\left(\mathcal{F}(U) \rightrightarrows \mathcal{F}(R)\right)$$
$$\coprod \mathcal{F}(U_i) \rightrightarrows \coprod \mathcal{F}(U_i \cap U_j)$$

ex: $\mathcal{O}_X \in X_{ét}$ via sheaf $\mathcal{O}_X'$ on $X_{ét'}$

via $\mathcal{O}_X'(T) = \mathcal{O}_T(T)$

sheaf on sites $(Sch/S)_{ét} \supset (Sch/X)_{ét'}$ so gives smthg in

$X_{ét'} \rightsquigarrow X_{ét}$

How to define sheaves on X? $R \overset{s}{\underset{t}{\rightrightarrows}} U \to X$

= sheaf $\mathcal{F}$ on U

together w/ $\varepsilon : s^*\mathcal{F} \to t^*\mathcal{F}$ " i.e. $U \to X$ effective descent."

s.t. " $\pi_{23}^*\varepsilon \; \pi_{12}^*\varepsilon = \pi_{13}^*\varepsilon$

on $U \times_X U \times_X U$.

why: this gives sheaves on $Et'(X)$ via descent.

$$\begin{tikzcd}
\forall\ T \arrow[r] & X \text{ scheme} \\
U \times_X T \arrow[r] \arrow[u, "ét"] & U \arrow[u] \\
R \times_X T \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & R = U \times_X U \arrow[u, shift left] \arrow[u, shift right] \\
R' \times T \arrow[r] \arrow[u] \arrow[u, shift left=2] \arrow[u, shift right=2] & R' = U \times_X U \times_X U \arrow[u] \arrow[u, shift left=2] \arrow[u, shift right=2]
\end{tikzcd}$$

$R \times_X T = (U \times_X T) \times_T (U \times_X T)$

Punchline: shf on X = shf on $Et'(X)$
$(Et(X))$ = desc. data for sheaf w/r/t to $U \to X$

Def: an $\mathcal{O}_X$ module M on $X_{ét}$ is quasi-coherent if $\exists\, U \to X$ étale surjection $(\forall)$
s.t. M_U is q. coherent over U.

Def Let X be a loc. Noeth. alg. space / S. An $\mathcal{O}_X$ mod M on $X_{ét}$ is coherent if $\exists\, U \to X$ étale surjection s.t. M_U coh. on U. $(\forall)$

Rem If $f: X \to Y$ is a morph. of alg. spaces, get induced morphism of topoi $f: X_{ét} \to Y_{ét}$

via: $Et(Y) \to Et(X)$
$(U \to Y) \mapsto (U \times_Y X \to X)$

$$f^* M = f^{-1} M \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$$

$f_* M$ induced by $\mathcal{O}_Y \to f_* \mathcal{O}_X$

Prop 7.1.9 If $f: X \to Y$ morph. of alg spaces / S
a) M q. coh on $Y \Rightarrow f^* M$ q. coh on X
b) f. q-compact, q-sep, N q. coh on $X \Rightarrow f_*$ q. coh on Y

Affine Morphisms

... commutative $\mathcal{O}_X$-algebras,

X alg. space, $\mathcal{A}$ q. coh. sheaf of commutative $\mathcal{O}_X$-algebras,

define $\mathrm{Spec}_X \mathcal{A}$ via

$$\mathrm{Spec}_X(\mathcal{A})(T) = \{ (f, \varepsilon) \mid f : T \to X,\ f^*\mathcal{A} \xrightarrow{\varepsilon} \mathcal{O}_T \}$$

$\mathcal{O}_T$-alg. morph

$$\begin{tikzcd} & \mathrm{Spec}_X(\mathcal{A}) \arrow[d] & \mathrm{Spec}(\mathcal{A}(U)) \arrow[l, hook'] \arrow[d] \\ T \arrow[ur] \arrow[r] & X & U \text{ affine} \arrow[l, hook'] \end{tikzcd}$$

<u>Prop</u>: $\mathrm{Spec}_X(\mathcal{A})$ is an algebraic space & the natural morphism $\mathrm{Spec}_X(\mathcal{A}) \to X$ is affine.

eq. of cat.s

$$\{ \text{affine morphisms } Y \to X \}$$
$$\updownarrow$$
$$\{ \text{q. coh. sheaves of } \mathcal{O}_X\text{-algebras } \mathcal{A} \}$$

<u>Applications</u>:

- Max'l reduced subspace

  X alg. spaces $\mathcal{N}_X \subset \mathcal{O}_X$ subsheaf of locally nilp. functions is a sheaf of ideals, $X_{red} = \mathrm{Spec}_X(\mathcal{O}_X / \mathcal{N}_X)$

  q. coh.

- Scheme-theoretic closure

$f : X \to Y$ alg spaces

$\mathrm{Spec}_Y(\mathcal{O}_Y/K)$ $\quad\quad K = \ker(\mathcal{O}_Y \to f_* \mathcal{O}_X)$

fcns on Y, which vanish on X

"scheme-theoretic closure of X in Y"