Lecture 2 : properties of morphisms

Thursday, August 21, 2014 11:05 AM

Reccomend: Review Hartshorne Ex 5.17

Smooth Morphisms, Differentials

Properties of morphism

$\varphi : A \to B$ comm. rings

We say φ is

- finite type if $\exists$ surjection $A[x_1, \ldots, x_n] \to B$
- finite presentation if $\exists$ surj $A[x_1, \ldots, x_n] \xrightarrow{\psi} B$ s.t. $\ker \psi$ finitely gen.

If $f : X \to Y$ morph. of schemes is

- quasi compact if $\forall$ affines $U \subset Y$, $f^{-1}(U)$ q. comp.
- quasi separated if $\Delta : X \to X \times_Y X$ is q. compact
- separated if $\Delta : X \to X \times_Y X$ is a closed immersion
  $\Rightarrow$ in inv. im of affines, $\cap$'s of affines are affines

$\Leftrightarrow$ in $f^{-1}(U)$ U affine, intersections of affines are q. compact

Practice in Hartshorne: Ex II 3.2, 3.3

Practice in Hartshorne: Ex II 3.2, 3.3

<u>Def</u>: $f: X \to Y$ loc. finite type if $\forall x \in X$ $\exists$

Affines $\begin{array}{c} \operatorname{Spec} B \ni x \\ \cap \\ X \end{array} \longrightarrow \begin{array}{c} \operatorname{Spec} A \ni f x \\ \cap \\ Y \end{array}$ s.t. $A \xrightarrow{f^\#} B$ is finite type

<u>Def</u>: $f: X \to Y$ loc. finite present. if $\forall x \in X$ $\exists$

nes $\begin{array}{c} \operatorname{Spec} B \ni x \\ \cap \\ X \end{array} \longrightarrow \begin{array}{c} \operatorname{Spec} A \ni f x \\ \cap \\ Y \end{array}$ s.t. $A \xrightarrow{f^\#} B$ is finite pres.

Aff

<u>Def</u>. $f: X \to Y$ finite type if f is loc. finite type & quasicompact.

see Hart Ex II 3.3

<u>Def</u>: $f: X \to Y$ finite pres if f is loc. finite pres & quasicompact & quasi-separated.

A commutative ring, M an A-mod

M is of finite presentation if $\exists$ right exact seq

$$A^r \to A^s \to M \to 0$$

[illegible] sheaf is loc. finitely

X a scheme $\mathcal{F}$ q.coh sheaf is ...
presented if $\forall$ affines $U \subset X$ ("Spec A"), $\Gamma(U, \mathcal{F})$ is a f.pres. A-mod.

Étale, smooth & unramified morphisms

$f : X \to Y$ is called

formally smooth (resp. frm. unram / form étale)
if $\forall\, Y' \to Y$, Y' affine, $Y'_0 \subset Y'$ defined by
a nilpotent ideal, the map

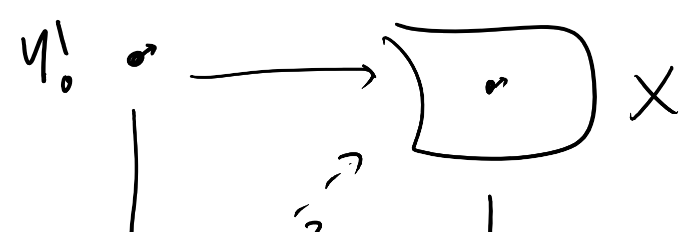
$Y' = \operatorname{Spec} A$, $Y'_0 = \operatorname{Spec} A/I$, $I^n = 0$ some n.

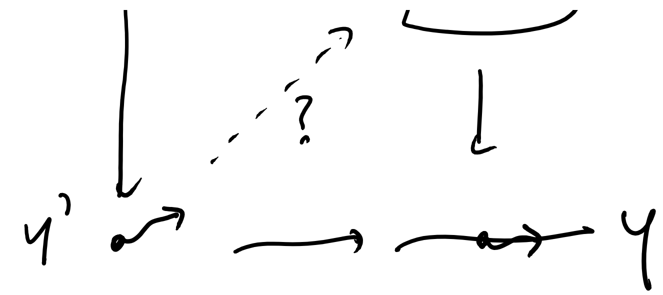
$$\mathrm{Hom}_Y(Y', X) \longrightarrow \mathrm{Hom}_Y(Y'_0, X) \text{ is}$$

surjective (resp injective / bijective)

We imagine Y' as a thickening of Y'_0 s.t. Y' can be retracted topologically back onto Y'_0

i.e. "$Y' = Y'_0 \times [0,1]$"
$\uparrow$ infinitesimal





formally smooth $\Rightarrow \exists$?

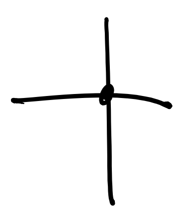
-"- unram $\Rightarrow \exists$ at most 1 ?

-"- étale $\Rightarrow \exists!$?

Def f is smooth (resp unram/étale) if f is formally sm. (unr./ét) & f is locally of finite pres.

k - field $\quad k[\varepsilon]/\varepsilon^2 \quad \operatorname{Spec} k[\varepsilon]/\varepsilon^2 = \mathbb{T}^1$

$X/k \quad (\mathbb{T}^1 \to X) \leftrightarrow \{(x,v) \mid x \in X, v \in T_x X\}$



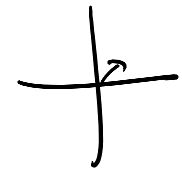
$xy = 0 \qquad \dfrac{k[x,y]}{xy}$

$k[x,y] \to k[\varepsilon]$

$x \mapsto \varepsilon$

$y \mapsto \varepsilon$

factors $\dfrac{k[x,y]}{xy} \to \dfrac{k[\varepsilon]}{\varepsilon^2}$



$\operatorname{Spec} k[\varepsilon] = Y'_0$

$$\nearrow \quad \operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^2} = Y'_0$$

$$\nearrow \quad \operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^3} = Y'$$

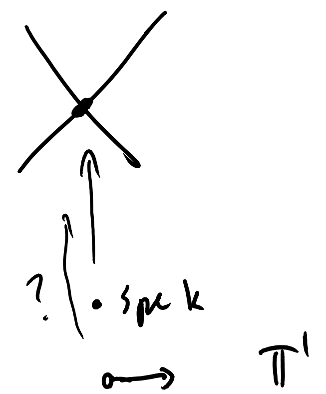
extends?

$$\frac{k[x,y]}{xy} \xrightarrow{?} \frac{k[\varepsilon]}{\varepsilon^3} \to \frac{k[\varepsilon]}{\varepsilon^2}$$

$$x \longmapsto \varepsilon + \lambda\varepsilon^2$$

$$y \longmapsto \varepsilon + \mu\varepsilon^2$$

no way!, $xy = \varepsilon^2 \neq 0$



extends, non uniquely $\Rightarrow$ not unram.

<u>Remark</u>: étale for $\mathbb{C}$ varieties = covering space map.

<u>Rem</u>: $Y' = \operatorname{Spec} A \qquad Y'_0 = \operatorname{Spec} A/I \qquad I^n = 0$

$$A \to A/I^{n-1} \to A/I^{n-2} \to \cdots \to A/I$$

$$A/I^{i+1} \to A/I^i$$

$\overline{I^i}$ is square 0 in A/I^{i+1}

So can always reduce to square 0 stuff.