

Stein Factorization $X, Y \ni$ alg. spaces

Def $f: X \rightarrow Y$ \mathbb{Z} -compact, \mathbb{Z} -sep. morphism, we say f is Stein if $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is an isom. of sheaves.

(ex: proper morph of varieties then Stein \Leftrightarrow fibers are connected (Zariski))
 \uparrow
 over $k = \bar{k}$

Main Observation: $f: X \rightarrow Y$ \mathbb{Z} -comp., \mathbb{Z} -sep then \exists a natural factorization

$$\begin{array}{ccccc}
 X & \xrightarrow{a} & X' & \xrightarrow{b} & Y \\
 & & \searrow & \nearrow & \\
 & & & f &
 \end{array}$$

a Stein
 b affine

$$X' = \text{Spec}_Y (f_* \mathcal{O}_X)$$

pt. of a product $\begin{matrix} X \\ \text{Spec } F \rightarrow X \end{matrix}$

ex: application:

$f: X \rightarrow Y$ sep. \mathbb{Z} -finite.

Stein factorization

$$\begin{array}{ccccc}
 X & \xrightarrow{g} & Z & \xrightarrow{h} & Y \\
 & & \searrow & \nearrow & \\
 & & & f &
 \end{array}$$

$\Rightarrow g$ open

$\Rightarrow f, g$ affine.

Theorem (Chow's Lemma)

S noeth scheme $X \xrightarrow{f} S$ sep morph. finite type
 X reduced alg. space. Then $\exists X' \rightarrow X$ proper birat'l
s.t. X' is a g. projective S -scheme.

Theorem (Finiteness of Cohomology)

$f: X \rightarrow Y$ proper morphism of loc. noeth. alg spaces
 $\{F_i\}$ coherent sheaf on X , then $R^q f_* F_i$ are coherent
on $Y \forall q \geq 0$.

(Recall $Y = \text{Spec } k$ $R^q f_* F$ a k -vector space, called $H^q(X, F)$
coherent $\Leftrightarrow H^q(X, F)$ finite dimensional)

Proof by induction on top space $|X|$

Omitting Representability stuff

e.g. "Algebraization of formal moduli" etc.
Artin

Algebraic Stacks!

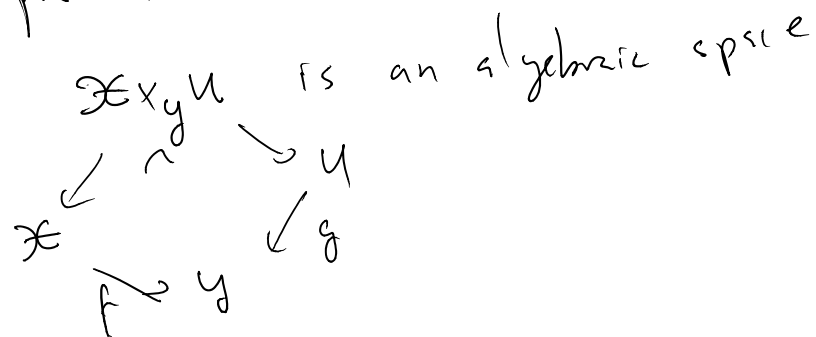
Recall: Given a groupoid in schemes $X_1 \rightrightarrows X_0$ can form an associated stack $[X_0/X_1]$ (over the big étale site).

Def² If $X_1 \xrightarrow[s]{t} X_0$ groupoid in $\text{Alg Spaces}/S$ w/ s, t smooth then can form $[X_0/X_1]$ = étale stackification of representable fibred cat in groupoids $\{X_0/X_1\}$

$\{X_0/X_1\}(T)$ obj $X_0(T)$
arrows $X_1(T)$

(Artin)
An alg^v stack is a stack equiv. to one as above.

Def A morphism $f: X \rightarrow Y$ of stacks over $\text{Et}(S)$ is called representable if $\forall U \in \text{Sch}/S$ and $g: U \rightarrow Y$ the fiber product



Lemma $f: X \rightarrow Y$ as above is representable \Leftrightarrow

$\nexists V \rightarrow Y$, V alg. space, $X \times_Y V$ an alg. space.

Pf: $\Leftarrow \checkmark$
 $\Rightarrow ?$

given $V \rightarrow Y$, cover $U \xrightarrow{g} V$ U scheme
 g étale

$$\begin{array}{ccccc}
 & \swarrow \text{space} & & & \\
 X \times_Y U & \rightarrow & X \times_Y V & \rightarrow & X \\
 \downarrow & & \downarrow & & \downarrow \\
 U & \rightarrow & V & \rightarrow & Y
 \end{array}$$