Lecture 20: from algebraic spaces to algebraic stacks

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Stein factorization $\quad X, Y$ alg. spaces

Def $f: X \to Y$ q-compact, q-sep. morphism, we say f is Stein if $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isom. of sheaves.

(ex: proper morph of varieties then Stein $\Leftrightarrow$ fibers are connected)
over $k = \bar{k}$ (Zariski)

Main Observation: $f: X \to Y$ q-comp, q-sep then $\exists$ a natural factorization

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[rr, bend right, "f"'] & X' \arrow[r, "b"] & Y \end{tikzcd}$$

a Stein
b affine

$$X' = \mathrm{Spec}_Y(f_* \mathcal{O}_X)$$

pt of a product $\overset{X}{=}$ $\mathrm{Spec}\, F \to X$

ex: applications:

$f: X \to Y$ sep. q-finite.

Stein factorization $\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend right, "f"'] & Z \arrow[r, "h"] & Y \end{tikzcd}$

$\Rightarrow$ g open

$\Rightarrow$ & f q. affine.

Theorem (Chow's lemma)

S noeth scheme $X \xrightarrow{f} S$ sep morph. finite type

X reduced alg. space. Then $\exists\ X' \to X$ proper birat'l

s.t. X' is a q. projective S-scheme.

Theorem (Finiteness of Cohomology) $f: X \to Y$

If $f: X \to Y$ proper morphism of loc. noeth. alg spaces

& F coherent sheaf on X, then $R^q f_* F$ are coherent

on Y $\forall\ q \geq 0$.

(Recall $Y = \operatorname{Spec} k$ $R^q f_* F$ a k-vector space, called $H^q(X, F)$

coherent $\Leftrightarrow$ $H^q(X, F)$ finite dimensional)

Proof by induction on top space $|X|$

Omitting Representability stuff

e.g. "Algebraization of formal moduli", etc.

Artin

Algebraic Stacks!

<u>Recall:</u> Given a groupoid in schemes/S $X_1 \rightrightarrows X_0$ can form an associated stack $[X_0/X_1]$ (over the big étale site).

<u>Def?</u> If $X_1 \underset{s}{\overset{t}{\rightrightarrows}} X_0$ gpoid in AlgSpaces/S w/ s,t smooth then can form $[X_0/X_1]$ = étale stackification of representable fibred cat in groupoids $\{X_0/X_1\}$

$\{X_0/X_1\}(T)$ objs $X_0(T)$
arrows $X_1(T)$

An (Artin) alg stack is a stack equiv. to one as above.

<u>Def</u> A morphism $f: \mathcal{X} \to \mathcal{Y}$ of stacks over $Et(S)$ is called representable if $\forall\, U \in Sch/S$ and $g: U \to \mathcal{Y}$ the fiber product

$\mathcal{X} \times_{\mathcal{Y}} U$ is an algebraic space

$$\begin{tikzcd}[column sep=small] & \mathcal{X}\times_{\mathcal{Y}} U \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\sim"] & \\ \mathcal{X} \arrow[dr, "f"'] & & U \arrow[dl, "g"] \\ & \mathcal{Y} & \end{tikzcd}$$

Lemma $f : \mathcal{X} \to \mathcal{Y}$ as above is representable $\iff$

$\forall\, V \to \mathcal{Y}$, V alg. space, $\mathcal{X} \times_{\mathcal{Y}} V$ an alg. space.

Pf: $\Leftarrow$ ✓

$\Rightarrow$? given $V \to \mathcal{Y}$, cover $U \xrightarrow[g]{\text{ét}} V$ U a scheme, g étale

$$\begin{array}{ccccc} \overset{\text{space}}{\mathcal{X}\times_{\mathcal{Y}} U} & \to & \mathcal{X}\times_{\mathcal{Y}} V & \to & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ U & \to & V & \to & \mathcal{Y} \end{array}$$