Lecture 21: Stacks, spaces and fiber products

Tuesday, October 28, 2014 11:03 AM

Ex 5.6:

Y/S alg. space, $F \to Y$ étale sheaf mapping to Y

then F is an alg space if and only if $\exists\ U \to Y$ étale

s.t. $F \times_Y U$ an alg space.

<u>Def</u> A stack $\mathcal{X}/S$ is an algebraic (Artin) stack if

1) $\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ is representable
2) $\exists$ smooth surjective morphism $\pi : X \longrightarrow \mathcal{X}$ w/ X a scheme.

<u>Recall:</u>

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"'] & \mathcal{Z} \end{tikzcd}$$

morph. of stacks $/S$ (étale)

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} = \mathcal{X} \underset{f,\mathcal{Z},g}{\times} \mathcal{Y}$$

our $T \to S \rightsquigarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}(T) =$ obj $\left(x \in \mathcal{X}(T),\ y \in \mathcal{Y}(T),\ \varphi : f(x) \xrightarrow{\sim} g(y) \right)$

morphs (comm. diagrams)

<u>2-Yoneda:</u> If $V, U \in Sch/S$, $\mathcal{X}/S$ stack then

$$\mathrm{Hom}(U, \mathcal{X}) \xrightarrow{\sim \text{ eq.}} \mathcal{X}(U)$$

$(\pi : U \to S)$ $\quad f \longmapsto f(\mathrm{id}_U)$

$U \longleftrightarrow (Sch/S)/U$

$\quad = Sch/U \quad T \to U$

$\quad\ \vdots$

$\quad Sch/S \quad T \to U \xrightarrow{\pi} S$

Sheaf avatar:

$U \longleftrightarrow (T \mapsto \mathrm{Hom}(T,U))$

$$U \xrightarrow{f} V \xrightarrow{\alpha} \mathcal{X}$$

$$\begin{array}{ccc} \alpha \in \mathrm{Hom}(V,\mathcal{X}) & \longrightarrow & \mathrm{Hom}(U,\mathcal{X}) \\ \downarrow & & \downarrow \\ \mathcal{X}(V) & & \mathcal{X}(U) \end{array}$$

comm in the sense that

$$\begin{array}{ccc} \alpha & \longrightarrow & \alpha \circ f \\ \downarrow & & \downarrow \\ X_\alpha & & X_{\alpha\circ f} \end{array}$$

is a choice for $f^* X_\alpha$

Main observation about the diagonal:

Given $t: T \to \mathcal{X}$, $u: U \to \mathcal{X}$ schemes, $\mathcal{X}$ stack then we have an equivalence of fib. cats (stacks)

$$U \times_{u,\mathcal{X},t} T \simeq (U \times_S T) \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$$

$$\begin{array}{ccc} & & U \\ & & \downarrow u \\ T & \xrightarrow{t} & \mathcal{X} \end{array} \qquad \begin{array}{ccc} & & U \times T \\ & & \downarrow (u,t) \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X}\times\mathcal{X} \end{array}$$

$(f, g, \varphi: f^* x_U \xrightarrow{\sim} g^* x_T)$

$(U \times_{\mathcal{X}} T)(V)$

$$\begin{array}{ccc} V & \xrightarrow{f} & U \\ \downarrow g & & \downarrow u \\ T & \xrightarrow{t} & \mathcal{X} \end{array}$$

2-yoneda

$u \longleftrightarrow x_U \in \mathcal{X}(U)$

$t \longleftrightarrow x_T \in \mathcal{X}(T)$

$$\begin{tikzcd} & U \\ V \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & T \end{tikzcd}$$

$(\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} U \times T)(V)$

$(x_V \in \mathcal{X}(V),\ f, g,\ \psi : (x_V, x_V) \xrightarrow{\sim} (f^*x_U, g^*x_T))$

$\mathcal{X}(V) \times \mathcal{X}(V)$

$(f, g, \varphi : f^* x_U \xrightarrow{\sim} g^* x_T) \longleftarrow (f, g, x_V, \psi : (x_V, x_V) \xrightarrow{\sim} (f^* x_U, g^* x_T))$

$$\begin{tikzcd} & g^* x_T \\ x_V \arrow[ur] \arrow[dr] & \\ & f^* x_U \end{tikzcd}$$

<u>Cor:</u> If $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ rep. ($\mathcal{X}/S$ stack)

then so is any morphism $U \to \mathcal{X}$ U scheme.

<u>Recall:</u> $(Sch/S)_{ét}$ $(AlgSpaces/S)_{ét}$

$\swarrow$ eq. of topoi

$\|$

$(AlgSpaces'/S)_{ét}$

$Ét(X) \overset{\text{eq. of topoi}}{\longleftarrow\!\!-\!\!\to} Ét'(X)$

X alg sp.   rest. dom to schemes

So $Sheaves/S = Shvs/(Alg.Spaces/S)_{ét}$

$\cong$

$Shv/(Sch/S)_{ét}$

$Stacks/(ASpaces/S)_{ét}$

eq.

$Stacks/(Sch/S)_{ét}$

In particular, 2-Yoneda:

Y alg. space/S $\quad \mathcal{X}$ stack

$$\mathrm{Hom}(Y, \mathcal{X}) \xrightarrow{\sim} \mathcal{X}(Y)$$

Diagonal Interpretation

$\mathcal{X}/S$ stack $\quad Y/S$ alg. space, $\quad f, g : Y \to \mathcal{X}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \updownarrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_f, x_g \in \mathcal{X}(Y)$

we have a fiber prod diagram of stacks

$$\begin{tikzcd} \mathrm{Isom}(x_f, x_g) \arrow[r] \arrow[d] & Y \arrow[d, "f,g"] \\ \mathcal{X} \arrow[r, "\Delta"'] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

sheaf.

fiber prod(V) : $\quad V \xrightarrow{\alpha} Y$

$V \xrightarrow{\beta} \mathcal{X}$

$x_\beta \in \mathcal{X}(V)$

$$(\alpha : V \to Y, x_\beta, \varphi : (\alpha^* x_f, \alpha^* x_g) \xrightarrow{\sim} (x_\beta, x_\beta))$$

$$\updownarrow$$

$$(\alpha : V \to Y, \psi : \alpha^* x_f \to \alpha^* x_g)$$

$$\mathrm{Isom}(x_f, x_g)(V)_{\alpha : V \to Y}$$