

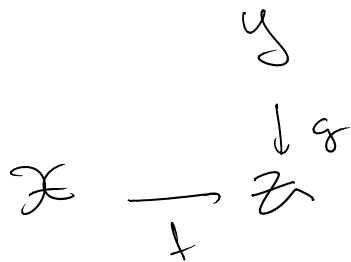
Ex 5.6:

$Y/S$  alg. space,  $F \rightarrow Y$  étale sheaf mapping to  $Y$   
 then  $F$  is an alg space if and only if  $\exists U \rightarrow Y$  étale  
 s.t.  $F \times_Y U$  an alg space.

Def A stack  $\mathcal{X}/S$  is an algebraic (Artin) stack if

- 1)  $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is representable
- 2)  $\exists$  smooth  $\uparrow$  morphism  $\pi: X \rightarrow \mathcal{X}$  s.t.  $X$  a scheme.  
 surjective

Recall:



morph. of stacks/S (étale)

$$\mathcal{X} \times_{\mathcal{Z}} Y = \mathcal{X} \times_{\mathcal{Z}, g}^X Y$$

$$\text{over } T \rightarrow S \rightsquigarrow \mathcal{X} \times_{\mathcal{Z}} Y(T) = \text{obj } \left( \begin{array}{l} (x \in \mathcal{X}(T), y \in Y(T), \\ \varphi: f(x) \xrightarrow{\sim} g(y)) \end{array} \right)$$

morphs (comm. diagrams)

2-Yoneda: If  $U, V \in \text{Sch}/S$ ,  $\mathcal{X}/S$  stack then

$$\text{Hom}(U, \mathcal{X}) \xrightarrow{\sim} \mathcal{X}(U)$$

$$(\pi: U \rightarrow S) \xrightarrow{f} f(\text{id}_U)$$

$$U \leftrightarrow \begin{array}{c} (\text{Sch}/S)/U \\ \text{Sch}/U \quad T \rightarrow U \\ \vdots \\ \text{Sch}/S \quad T \rightarrow U \end{array}$$

$$u \xrightarrow{f} v \xrightarrow{\alpha} \mathcal{E}$$

Sheaf version:  
 $u \longleftarrow (T \rightarrow \text{Hom}(T, u))$

$$\begin{array}{ccc} \alpha \in \text{Hom}(v, \mathcal{E}) & \longrightarrow & \text{Hom}(u, \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{E}(v) & & \mathcal{E}(u) \end{array}$$

comm in the sense that

$$\begin{array}{ccc} \alpha & \longrightarrow & \alpha \circ f \\ \downarrow & & \downarrow \\ X_\alpha & & X_{\alpha \circ f} \end{array}$$

is a choice for  $f^* X_\alpha$

Main observation about the diagram:

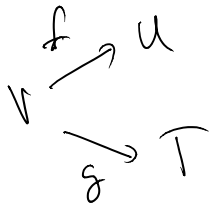
Given  $t: T \rightarrow \mathcal{E}$ ,  $u: U \rightarrow \mathcal{E}$  schemes,  $\mathcal{E}$  stack  
 then we have an equivalence of fibrations (stacks)

$$U \times_{u, \mathcal{E}, t} T \cong (U \times_S T) \times_{\mathcal{E} \times \mathcal{E}} \mathcal{E}$$

$$\begin{array}{ccc} U & & U \times T \\ \downarrow u & & \downarrow (u, t) \\ T \xrightarrow{t} \mathcal{E} & \xrightarrow{\Delta} & \mathcal{E} \times \mathcal{E} \end{array}$$

$$\begin{array}{ccc} (f, g, \varphi: f^* X_u \cong g^* X_T) & & \\ \downarrow & & \downarrow \\ (U \times_{\mathcal{E}} T)(v) & \xrightarrow{f} & U \\ \downarrow & & \downarrow \\ T \xrightarrow{t} \mathcal{E} & & \mathcal{E} \end{array}$$

2-yoneda  
 $u \longleftarrow X_u \in \mathcal{E}(u)$   
 $t \longleftarrow X_T \in \mathcal{E}(T)$



$$(\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} U \times T)(V) \\
 (x, y \in \mathcal{X}(V), f, g, \psi: (x, y) \xrightarrow{\sim} (f^*x, g^*y)) \\
 \mathcal{X}(V) \times \mathcal{X}(V)$$

$$(f, g, \varphi: f^*x_u \xrightarrow{\sim} g^*x_T) \longleftarrow (f, g, x_v, \psi: (x_v, x_v) \xrightarrow{\sim} (f^*x_u, g^*x_T)) \\
 \begin{array}{ccc}
 & & g^*x_T \\
 x_v & \rightarrow & \\
 & \searrow & \\
 & & f^*x_u
 \end{array}$$

Cor: If  $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is a monomorphism without loss of generality then so is any morphism  $u: \mathcal{X} \rightarrow \mathcal{X}$  a scheme.

Recall:

$$(\text{Sch}/S)_{\text{ét}} \cong (\text{Alg Spaces}/S)_{\text{ét}}$$

$\uparrow$  eq. of topoi

$$\text{Et}(X) \xrightarrow{\text{eq. of topoi}} \text{Et}'(X) \\
 X \text{ alg sp.} \quad \text{nest. dom to schemes}$$

$$\text{So } \text{Schemes}/S = \text{Shvs} / ((\text{Alg. Spaces}/S)_{\text{ét}}) \\
 \cong \text{Shvs} / (\text{Sch}/S)_{\text{ét}}$$

$$\text{Stacks}/(\text{Sch}/S)_{\text{ét}} \xrightarrow{\text{eq.}} \text{Stacks}/(\text{Alg Spaces}/S)_{\text{ét}}$$

In particular, 2-Yoneda:

$$Y \text{ alg. specs} \not\cong \text{stack}$$

$$\text{Hom}(Y, \mathcal{X}) \xrightarrow{\sim} \mathcal{X}(Y)$$

Diagonal interpretation

$\mathcal{X}/S$  stack  $Y/S$  alg-spec.,  $f, g: Y \rightarrow \mathcal{X}$   
 $\downarrow$   
 $X_f, X_g \in \mathcal{X}(Y)$   
 we have a fiber prod diagram of stacks

$$\begin{array}{ccc} \text{Isom}(X_f, X_g) & \longrightarrow & Y \\ & & \downarrow f, g \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X} \end{array}$$

sheaf.

fiber prod(V) :

$$\begin{array}{ccc} V \xrightarrow{\alpha} Y & & (\alpha: V \rightarrow Y, X_\beta, \psi: (\alpha^* X_f, \alpha^* X_g)) \\ V \xrightarrow{\beta} \mathcal{X} & & \downarrow \\ & & (X_\beta, X_\beta) \end{array}$$

$X_\beta \in \mathcal{X}(V)$

$$\begin{array}{ccc} (\alpha: V \rightarrow Y, \psi: \alpha^* X_f \rightarrow \alpha^* X_g) & & \\ \text{Isom}(X_f, X_g)_{\alpha: V \rightarrow Y} & & \end{array}$$