Lecture 22: properties of stacks and morphisms of stacks

Tuesday, October 28, 2014 11:16 AM

Def'ns of alg. stacks $\longrightarrow$ $\mathcal{X}$ a stack s.t. • Δ rep. by alg. spaces
• $\exists\, X \to \mathcal{X}$ sm. surj. X alg. space

$\Updownarrow$ $\boxed{\text{Tag } 04T3}$

$\mathcal{X}$ = stackification of $\{X_0/X_1\}$ $\leftarrow$ rep. groupoid given by

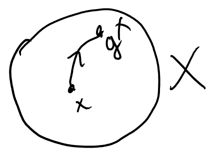
$X_1 \overset{t}{\underset{s}{\rightrightarrows}} X_0$ alg. spaces

s, t smooth

example: Stack Quotients

X/S algebraic space G/S smooth gp. scheme acting on X

$[X/G]$ "stack quotient" Def 1: $[X/G]$ = stackification of $\{X/G\}$



$$\boxed{G \times X \overset{\pi_2}{\underset{act}{\rightrightarrows}} X}$$

Def 2:

$[X/G]$ objects: triples (T, P, π)

$$\begin{array}{c} P \\ \downarrow \\ T \end{array}$$

P is a G_T-torsor on the étale site of T

$\downarrow$

$(Sch)_{ét}$

$\pi : P \longrightarrow X_T$ G_T equivariant

morphisms $(T, P, \pi) \longrightarrow (T', P', \pi')$ $(f, f^\flat)$

$T \xrightarrow{f} T'$

$P \xrightarrow{f^\flat} f^*P'$ s.t. comp. w/ π's.

$P \xrightarrow{\pi} X_T$

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "f^b"] & X_T \\ f^*P' \arrow[ur, "f^*\pi'"'] & \end{tikzcd}$$

This is a stack since it is a sheaf construction.

Δ is representable? i.e. $\underline{\mathrm{Isom}}_T((P,\pi),(P',\pi'))/_T$ space?

By going local & props of torsors, WLOG, can assume $P = P' = G_T$

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d] & X \\ P' \arrow[ur, "\pi'"'] & \end{tikzcd}$$

$$\begin{tikzcd} G_T \arrow[r, "\pi"] \arrow[d, "G(T) \ni g"'] & X_T \\ G_T \arrow[ur, "\pi'"'] & \end{tikzcd}$$

$$\begin{tikzcd} \text{space} = \mathrm{Isom} \arrow[r] \arrow[d] & G_T \arrow[d, "(\pi, \pi' \circ g)"] \\ X_T \arrow[r] & X_T \times X_T \end{tikzcd}$$

<u>ex:</u> G/S smooth gp scheme $BG = B_S G = [S/G]$ (triv. action)

"Classifying space"

by above $\mathrm{Hom}(T, BG) \longleftrightarrow$ Category of G_T torsors.

<u>Prop</u> if $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ alg. stacks$/S$

8.1.16 $\mathcal{X}$,

8.1.16

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \downarrow c \\ \mathcal{Y} & \xrightarrow{d} & \mathcal{Z} \end{array}$$

then $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is an algebraic stack.

Lemma

$$\begin{array}{ccccc} & & & & \mathcal{X} \\ & & & & \downarrow \\ \mathcal{Y}'' & \to & \mathcal{Y}' & \to & \mathcal{Y} \end{array}$$

Fibred cats in groupoids, then

$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}') \times_{\mathcal{Y}'} \mathcal{Y}'' \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}''$

equivalence

Pf sketch:

$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}') \times_{\mathcal{Y}'} \mathcal{Y}'' \longrightarrow (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'')$

$(x, y', \sigma, y'', \tau) \longmapsto (x, y'', x|_{\mathcal{Y}} \xrightarrow{\sim} y''|_{\mathcal{Y}}$

$\sigma : x|_{\mathcal{Y}} \xrightarrow{\sim} y'|_{\mathcal{Y}}$ $\quad \tau : y' \xrightarrow{\sim} y''|_{\mathcal{Y}'}$ $\quad \sigma \circ y'|_{\mathcal{Y}} \xrightarrow[\tau|_{\mathcal{Y}}]{} \overset{\shortparallel}{y''|_{\mathcal{Y}'}|_{\mathcal{Y}}}$

Proof of prop:

know

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \downarrow c \\ \mathcal{Y} & \xrightarrow{d} & \mathcal{Z} \end{array}$$

$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a stack

need to show: Δ rep., $\exists$ sm. cover

given objects $\alpha = (x, y, \sigma)$, $\underset{\beta}{\overset{\shortparallel}{(x', y', \sigma')}}$ in $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}(T)$

want to show $\mathrm{Isom}(\alpha, \beta)$ is an algebraic space.

// obj

$$\left\{ \begin{array}{c} x \xrightarrow{f} x' \\ y \xrightarrow[a]{} y' \end{array} \;\middle|\; \begin{array}{ccc} cx & \xrightarrow{f} & cx' \\ \sigma \downarrow & & \downarrow \sigma' \\ dy & \longrightarrow & dy' \end{array} \text{ commutes} \right\}$$

$$\left\{ y \xrightarrow[g]{} y' \;\middle|\; \begin{array}{ccc} \sigma\downarrow & & \downarrow\sigma' \\ dy & \xrightarrow[g]{} & dy' \end{array} \text{ commutes} \right\}$$

$\cap$

$$\begin{tikzcd}
\mathrm{Isom}(x,x') \times \mathrm{Isom}(y,y') \arrow[d] & \mathrm{Isom}(\alpha,\beta) \arrow[l] \arrow[d] \\
\mathrm{Isom}(cx,dy') \times \mathrm{Isom}(cx,dy') & \mathrm{Isom}(cx,dy') \arrow[l, "\Delta"']
\end{tikzcd}$$

(f, g)

$(\sigma' \circ cf, dg \circ \sigma)$

rep since Z algebraic

$$\begin{tikzcd}
(\;) \arrow[r, "sm.surj"] \arrow[d, "sm.surj"] & (\;) \arrow[r] \arrow[d, "sm.surj"] & X \arrow[d, "sm.surj"] \\
(\mathcal{X} \times_Z Y) \times_y Y \arrow[r, "sm.surj"] \arrow[d] & \mathcal{X} \times_Z Y \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
Y \arrow[r, "sm.surj"'] & y \arrow[r] & Z
\end{tikzcd}$$

alg.space

$\exists$ smooth cover?

Lemma pullback of sm. surj. is smooth surj.

$$\begin{tikzcd}
? \arrow[r] \arrow[d, "sm. surj ?"'] & \mathcal{X} \times_Z y \arrow[r] \arrow[d, "sm. surj"] & \mathcal{X} \arrow[d, "sm. surj"] \\
Y \arrow[r] \arrow[rr, bend right] & y \arrow[r] & Z
\end{tikzcd}$$

scheme

by previous lemma.

Def: If P is a property of S schemes, stable in the smooth topology, then we say an alg. stack has prop P if

$\exists\ X \xrightarrow{\pi} \mathfrak{X}$ π sm. surj., X a scheme w/ prop P.

$(\forall)$

<u>ex:</u> loc. noeth, regular, loc. finite type, loc. finite presentation.