Lecture 23: properties of morphisms of stacks, inertia and Deligne-Mumford stacks

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<u>Lemma/Excercise</u> $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ morph. of stacks on

$\mathcal{C}$ a site s.t. $\forall T \in \mathrm{ob}(\mathcal{C})$ $\exists$ cov'g $\{T_i \to T\}$ s.t.

$\mathcal{X}|_{T_i} \longrightarrow \mathcal{Y}|_{T_i}$ an equivalence all i $\Rightarrow$ f an equivalence.

Quot's by groups:

$X/_S$ alg. spce, $G/_S$ smooth gp scheme

$\{X/G\}$ cat. fibred in groupoids $\{X \times G \rightrightarrows X\}$

$\{X/G\} \longrightarrow [X/G]$ via torsors

$\{X/G\}(T)$

$G_T \longrightarrow X_T$

$e \longmapsto x \in X_T(T)$

objects: $X(T) \ni x$

morphisms $X(T) \times G(T)$

Spirit of why this is an equivalence: étale locally, any torsor is $\simeq G$

Properties of morphisms

If P is any prop. of alg spaces we can say a morphism $f: X \to Y$ of alg. stacks has prop P if f is representable by alg. spaces & if all spaces representing f have P.

$\quad T \quad \mathcal{X}$

i.e. all spaces representing :

i.e.

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Y}} T \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ T \arrow[r] & \mathcal{Y} \end{tikzcd}$$

have P

Given a morphism of stacks, $f : \mathcal{X} \to \mathcal{Y}$, a chart for f is space diagram of f X, Y, p, q, h as in diagram

i.e. if P is a ~~prop of~~ smooth ~~$\mathcal{X}$~~ morphisms ~~, stable~~ alg sp. loc. on domain

repr. + smooth top, $\downarrow$ smooth of alg spaces, p. smooth

then we say f (as above) ~~has~~ has P $\iff$ h has P.

Prop: This is independent of charts.

Pf: common refinements.

Inertia: $\mathcal{X}/S$ an algebraic stack, $I_{\mathcal{X}}$

$\downarrow$ rep. in alg. spaces

$\mathcal{X}$

$(x, g) \rightsquigarrow$

$$\begin{tikzcd} I_{\mathcal{X}}|_T \arrow[r] \arrow[d] & I_{\mathcal{X}} \arrow[d] \\ T \arrow[r] & \mathcal{X} \end{tikzcd}$$

$g \in \mathrm{Aut}(x)$

$I_{\mathcal{X}}$ objects (x, g) g aut of x

morphs comm. diagrams

$$\begin{tikzcd} T \arrow[r] & \mathcal{X} \end{tikzcd}$$

$x \in \mathcal{X}(T)$

$\mathcal{I}_{\mathcal{X}}$ morphs comm. diagrams

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"] & x' \arrow[d, "g'"] \\ x \arrow[r, "f"'] & x' \end{tikzcd}$$

$x, y,$

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"'] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

representable in spaces.

$$\begin{tikzcd} x, x \arrow[d, "\alpha"', bend right] \arrow[d] \arrow[d, "\beta", bend left] \\ y, y \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[d, "\alpha"'] & x \arrow[d, "\beta"] \\ y & y \end{tikzcd} \longrightarrow \left[\begin{tikzcd} x \arrow[r, bend right, "\beta^{-1}\alpha"'] & x \end{tikzcd}\right]$$

eq. of cats.

$$\begin{tikzcd} \underline{\mathrm{Aut}}(x) \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}} \arrow[d] \\ T \arrow[r, "x \in \mathcal{X}(T)"'] & \mathcal{X} \end{tikzcd}$$

$\mathcal{M}_g \quad g \geq 2 \quad$ stack objects$/T$

$\mathcal{M}_g(T)$ smooth morphisms

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"] \\ T \end{tikzcd}$$

fibers are genus g curves

$T = \mathrm{Spec}\, \mathbb{C}$

$$\begin{tikzcd} C \arrow[d] \\ T = \mathrm{Spec}\, \mathbb{C} \end{tikzcd}$$

"curve"

$$\begin{tikzcd} \mathcal{I}_{\mathcal{M}_g} \arrow[d] & \underline{\mathrm{Aut}}(C) \arrow[l] \arrow[d] \\ {} & {} \end{tikzcd}$$

$\mathcal{M}_g \underset{C}{\leftarrow} T$

$f : \mathcal{X} \to \mathcal{Y}$ alg stacks, then

$\Delta_{\mathcal{X}/\mathcal{Y}} : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable.

why? left to the reader.

$$\begin{tikzcd} \mathcal{X} \arrow[dr, "\text{rep?}"] & & \\ & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

$\Rightarrow$ can talk about f being q-compact, q-sep.

<u>Def</u>: An algebraic stack is Deligne-Mumford if $\exists$ étale surjection $X \to \mathcal{X}$ where X is a scheme.

<u>Thm</u> $\mathcal{X}/S$ alg. stack is DM $\Leftrightarrow$ $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is formally unramified.