

Lemma/Exercise $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ morph. of stacks on \mathcal{C} a site s.t. $\forall T \in \text{ob}(\mathcal{C}) \exists \text{ covering } \{T_i \rightarrow T\}$ s.t. $\mathcal{X}|_{T_i} \rightarrow \mathcal{Y}|_{T_i}$ an equivalence all $i \Rightarrow f$ an equivalence.

Quot's by groups

X/S alg-space, G/S smooth gp scheme

$\{X/G\}$ cat. fibred in groupoids $\{X \times G \rightrightarrows X\}$

$\{X/G\} \rightarrow [X/G] \leftarrow \text{via torsors}$

$\{X/G\}(T)$
 objects: $X(T) \times$
 morphisms $X(T) \times G(T)$
 $G_T \rightarrow X_T$
 $e \mapsto x \in X_T(T)$

Spirit of why this is an equivalence: étale locally, any torsor is $\cong G$

Properties of morphisms

If P is any prop. of alg spaces we can say a morphism of alg. stacks has prop P if f is representable, $f: X \rightarrow Y$
 if all spaces representing f have P .
 by alg. spaces

if all spaces represent

$$\begin{array}{ccc} \text{i.e.} & & \\ \text{have} & \rightarrow & \mathcal{X}_{x,y,T} \rightarrow \mathcal{X} \\ p & & \downarrow \quad \downarrow \\ & & T \rightarrow Y \end{array}$$

Given a morphism of stacks, $f: \mathcal{X} \rightarrow \mathcal{Y}$, a chart for f is X, Y, p, q, h as in

if P is a property of algebraic spaces, then we say f has P if P is a property of algebraic spaces, then we say f has P .

Prop 2: This is independent of charts.
Pf: common refinements.

Inverts \mathcal{X}/S an algebraic stack, \mathcal{X} indep. in alg. spaces

$$\begin{array}{ccc} (x,y) \sim & & \\ g \in \text{Aut}(x) & \sim & d_{\mathcal{X}}|_T \rightarrow d_{\mathcal{X}} \\ & & \downarrow \quad \downarrow \\ & & T \rightarrow \mathcal{X} \end{array}$$

objects (x,y) $g \in \text{Aut}(x)$
 maps $d_{\mathcal{X}}$
 com. diagrams

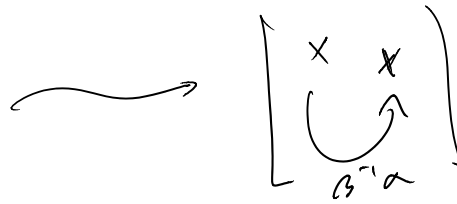
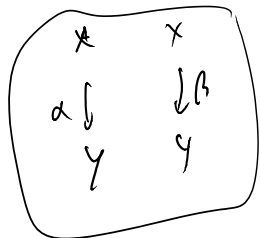
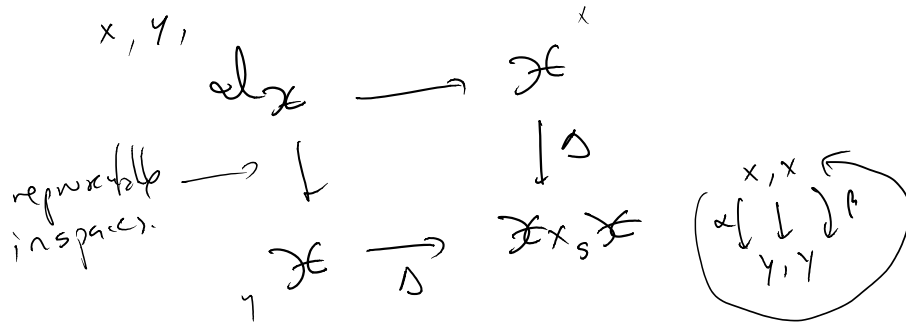
$$T \xrightarrow{\quad} \mathcal{X}$$

\uparrow
 $x \in \mathcal{X}(T)$

\mathcal{X}

maps
com. diagrams

$$\begin{array}{ccc} x & \xrightarrow{f} & x' \\ \downarrow g & & \downarrow g' \\ x & \xrightarrow{f} & x' \end{array}$$



eq. of cuts.

$$\begin{array}{ccc} \text{Aut}(x) & \rightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \rightarrow & \mathcal{X} \\ & & x \in \mathcal{X}(T) \end{array}$$

\mathcal{M}_g $g \geq 2$ stack objects/ T

$\mathcal{M}_g(T)$ smooth morphisms
 \mathcal{C}
fibers are genus g curves
 $\downarrow \pi$
 T

$T = \text{Spec } \mathbb{C}$

\mathcal{C} "cone"
 \downarrow
 $T = \text{Spec } \mathbb{C}$

$$\begin{array}{ccc} \mathcal{M}_g & \leftarrow & \text{Aut}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \cdot & & \cdot \end{array}$$

$$M_y \cong T$$

$f: X \rightarrow Y$ algebraic stacks, then

$\Delta_{X/Y}: X \rightarrow X \times_Y X$
is representable.

why? left to the reader.

$$\begin{array}{ccc} X & & \\ \downarrow \text{rep!} & & \\ X \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

\Rightarrow can talk about f by q -compact, q -sep.

Def An algebraic stack is Deligne-Mumford if \exists étale surjection $X \rightarrow \mathcal{X}$ where X is a scheme.

Thm \mathcal{X}/S alg-stack is DM $\Leftrightarrow \Delta: \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is formally unramified.