Lecture 24: Deligne-Mumford stacks and moduli of curves

Thursday, November 6, 2014 11:06 AM

<u>Recall</u>: An algebraic stack $\mathcal{X}/S$ is Deligne-Mumford if $\exists\ X \to \mathcal{X}$ étale surjection, with X a scheme (alg. space)

<u>Theorem</u>: $\mathcal{X}/S$ alg. stack is Deligne-Mumford if and only if the diagonal morphism $\Delta: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is formally unramified.

<u>Recall</u>: $X \xrightarrow{f} Y$ formally unram if $\forall\ Y'$ affine $Y' \to Y$ and any $Y'_0 \hookrightarrow Y'$ given by nilp. ideal,

$$\mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(Y'_0, X) \text{ is injective.}$$

<u>Part of sketch of part of proof</u>:

Suppose $\mathcal{X}$ is DM stack. why is Δ form. unram?

$X \xrightarrow{\pi} \mathcal{X}$ étale surjection

$$\begin{array}{ccc} X \times_{\mathcal{X}} X & \xrightarrow{b} & X \times_S X \\ a \downarrow \text{ét. surj.} & & \downarrow \text{étale surj.} \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times_S \mathcal{X} \end{array}$$

? $\left(x, y,\ \alpha \in \mathcal{X} \quad \pi x \xrightarrow{\sim} \alpha \xleftarrow{\sim} \pi y \right)$

Claim: b is unram. If it was, then the comp $\Delta \circ a$ is unram. $\Rightarrow \Delta$ unram.

EGA IV.17.1.3(v)

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$$X \times_{\mathcal{X}} X \xrightarrow{b} X \times_S X$$

$$\begin{array}{ccc} X \times_{\mathcal{X}} X & \xrightarrow{\text{étale}} & X \\ \downarrow \text{étale} & & \downarrow \text{étale} \\ X & \xrightarrow{a} & \mathcal{X} \end{array}$$

$$\begin{array}{ccc} X \times_{\mathcal{X}} X & \xrightarrow{\text{étale}} & X \\ & \text{unram} \searrow & \nearrow \\ & & X \times_S X \end{array}$$

unram $\Uparrow$ ega IV.17.3.5

Examples

Group quotients

X/S scheme G/S smooth gp scheme acting on X

$[X/G]$ stack quotient, then $[X/G]$ is DM if and only if $\forall s : \operatorname{Spec} k \to S$, $k = \bar{k}$ and $t \in [X/G](k)$, the stabilizer group $G_t \subset G_s$ is étale over $\operatorname{Spec} k$.

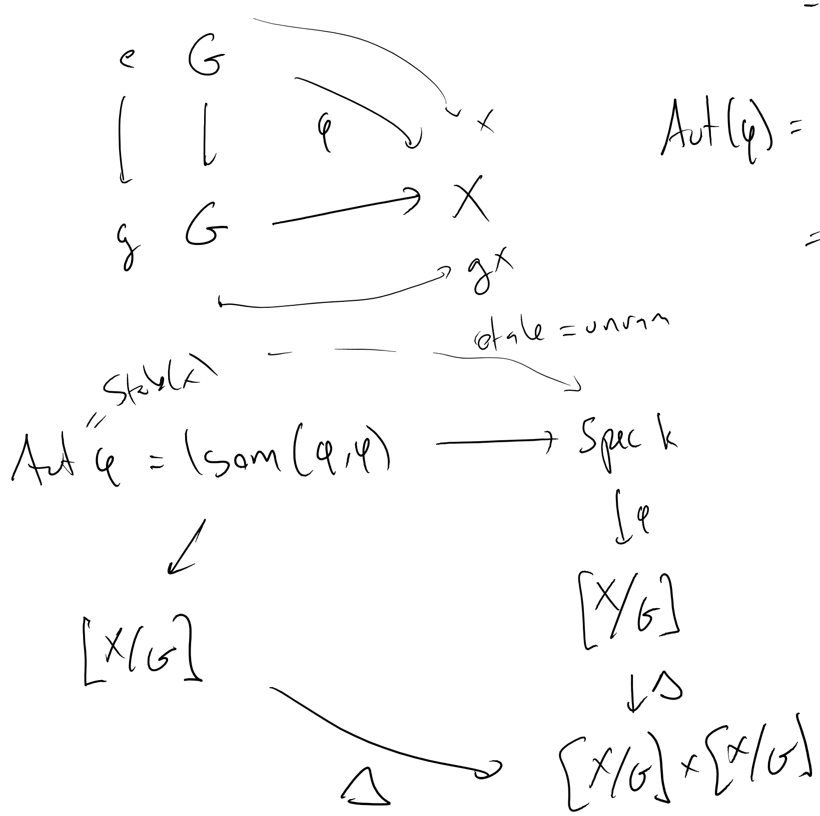
Why? unram can be checked on geom pts.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X} \times_S \mathcal{X} \\ & k \nearrow & \downarrow \\ & \searrow & S \end{array}$$

WLOG $S = \operatorname{Spec} k$. $k = \bar{k}$

G torsors $= G$ unram over $\operatorname{Spec} k$ = disjoint union of $\operatorname{Spec} k$'s.

= étale.



$\mathrm{Aut}(\varphi) = \{ g \in G \mid gx = x \}$

$= \mathrm{Stab}(x)$

$x = \varphi(e)$

$\mathcal{M}_g$ $g \geq 2$ fibred cat over Sch/S

objects $(T, f: C \to T)$ T/S scheme, f smooth proper morphism

$(g, \tilde{g}) : (T, f: C \to T) \to (T', f': C' \to T')$ s.t. geom. fibers are connected genus g curves.

is a cartesian square

$$\begin{array}{ccc} C & \xrightarrow{\tilde{g}} & C' \\ f \downarrow & & \downarrow f' \\ T & \xrightarrow[g]{} & T' \end{array}$$

Thm $\mathcal{M}_g$ is a DM stack.

Showed before that $\mathcal{M}_g$ is a stack. if C has genus ≥ 2 $\Omega_{C/T}$ ample

Basic idea: construct $\mathcal{M}_g$ as a quotient of a scheme by a gp w/ finite gp schemes as stabilizers.

Main observation: given $f: C \to T$, $L_{C/T} = \Omega_{C/T}^{\otimes 3}$

then $L_{C/S}$ is relatively v. ample, $f_* L_{C/T}$ loc. free rk $5g-5$

in particular, can obtain an embedding

$$C \longrightarrow \mathbb{P}\left(\left(f_* L_{C/T}\right)^\vee\right) \quad \text{"tri-canonical embedding"}$$

Define new fibered cat: $\widetilde{\mathcal{M}}_g$ objects $(f: C \to T, \sigma: \mathcal{O}_T^{5g-5} \xrightarrow{\sim} f_* L_{C/T})$