Lecture 25: Moduli of curves

Tuesday, November 11, 2014 11:15 AM

$\mathcal{M}_g(T) = \{ C \xrightarrow{f} T$ smooth proper, fibers connected curves genus $g \}$

$\tilde{\mathcal{M}}_g(T) = \{ C \xrightarrow{f} T$ as above, together w/ $\sigma : \mathcal{O}_T^{5g-5} \xrightarrow{\sim} f_* \mathcal{L}_{C/T} \}$

$\mathcal{L}_{C/T} = \omega_{C/T}^{\otimes 3}$

$\tilde{\mathcal{M}}_g(T) \longrightarrow \mathcal{M}_g(T)$ not surj. as presheaves, but as sheaves

Note, for a curve C (in fibers) as above, Hilbert poly is of the form

$$\underset{\substack{\uparrow \\ \text{degree}}}{(6g-6)}\, T + \underset{\substack{\uparrow \\ \text{via genus}}}{(1-g)}$$

$\mathcal{H}'$ = Hilbert scheme of subschemes of $\mathbb{P}^{5g-6}$ w/ Hilb. poly.

$\mathcal{H}$ = smooth curves in $\mathcal{H}'$ open subscheme.

$$\begin{tikzcd}
Z \arrow[r, "\text{closed}"] \arrow[dr, "\text{flat.}"'] & \mathcal{H}' \times \mathbb{P}^{5g-6} \arrow[d] & Z|_{\mathcal{H}} \arrow[d, "\text{smooth}"] \\
& \mathcal{H}' \arrow[r, phantom, "\supset"] & \mathcal{H}
\end{tikzcd}$$

$\tilde{\mathcal{M}}_g \longrightarrow \mathcal{H}$

$C \xrightarrow{f} T$

$C \longrightarrow \mathbb{P}_T((f_* \mathcal{L}_{C/T})^\vee) \to \mathbb{P}_T^{5g-6}$

$\sigma : \mathcal{O}^{5g-5} \xrightarrow{\sim} f_* L_{C/T}$

given $C \in \mathcal{H}(k=\bar{k})$ $\rightsquigarrow$ define $L'_{C/T} = \iota^* \mathcal{O}_{\mathbb{P}^{5g-6}}(1)$

$$\begin{tikzcd} C \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & \mathbb{P}^{5g-6} \arrow[d, "\pi"] \\ & T \end{tikzcd}$$

can check $L'_{C/T}$ same degree, global sections as $L_{C/T}$

$\tilde{\tilde{\mathcal{H}}}$ $\quad C,\ \iota : C \hookrightarrow \mathbb{P}^{5g-6}$

$$\begin{tikzcd} (\iota,\lambda) \arrow[r] & L'_{C/T} \arrow[r, "\sim", "\lambda"'] \arrow[d, squiggly] & L_{C/T} \\ \in \mathcal{H} & & \end{tikzcd}$$

$f_* L'_{C/T} = \pi_* \mathcal{O}_{\mathbb{P}^{5g-6}}(1) = \mathcal{O}_T^{5g-5}$

$\tilde{\tilde{\mathcal{H}}} \to \mathcal{H}$ need to show $\tilde{\tilde{\mathcal{H}}}$ representable.

Intermediate: $\tilde{\mathcal{H}}$ $\quad (\iota, \tilde{\lambda})$ $\quad \iota : C \hookrightarrow \mathbb{P}^{5g-6}$

$\tilde{\lambda} : f_* L'_{C/T} \xrightarrow{\sim} f_* L_{C/T}$

$\tilde{\mathcal{H}} \to \mathcal{H}$ is a GL_{5g-5} bundle

$\tilde{\mathcal{M}}_g \longrightarrow \tilde{\mathcal{H}}$

$$\begin{tikzcd} (C, \sigma : \mathcal{O}_T^{5g-5} \xrightarrow{\sim} f_* L_{C/T}) \arrow[r, mapsto] & C \arrow[r, hook] \arrow[rr, bend left, "\iota"] & \mathbb{P}((f_* L_{C/T})^\vee) \arrow[r, phantom, "\cong"] & \mathbb{P}^{5g-6} \end{tikzcd}$$

σ induces $\pi_* \mathcal{O}_{\mathbb{P}}(1) \xrightarrow{\sim} f_* L_{C/T}$

$$\begin{tikzcd} \pi_* \mathcal{O}_{\mathbb{P}}(1) \arrow[r, "\sim"] \arrow[d, equal] & f_* L_{C/T} \\ f_* L'_{C/T} \arrow[ur, "\tilde{\lambda}"'] & \end{tikzcd}$$

image of $\tilde{\mathcal{M}}_g$ is naturally $\tilde{\tilde{\mathcal{H}}} \underset{\text{closed}}{\hookrightarrow} \tilde{\mathcal{H}} \ldots$

$\Longrightarrow$ $\tilde{\mathcal{M}}_g$ represented by a scheme. (quasi-projective)

$$\begin{tikzcd}
\tilde{\mathcal{M}}_g \arrow[r] \arrow[d] & \mathcal{M}_g \arrow[dl, bend left] \\
{[\tilde{\mathcal{M}}_g / G]} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd} C \arrow[d, "f"] \\ T \end{tikzcd}$$

$T_i \hookrightarrow T$

$G = GL_{5g-5}$ acts on $\tilde{\mathcal{M}}_g$ via

$g\left(\begin{smallmatrix} C \\ \downarrow f \\ T \end{smallmatrix}, \sigma : \mathcal{O}^{5g-5} \xrightarrow{\sim} f_* L_{C/T}\right)$

via " $(C, \underline{\sigma^{-1} g})$?

$C, \sigma : \mathcal{O}^{5g-5} \xrightarrow{\sim} f_* L$

$$\begin{tikzcd} G \arrow[r] & \boxed{G_T \arrow[r] & (\tilde{\mathcal{M}}_g)_T} \end{tikzcd}$$

$$e \in G_T \longrightarrow (\tilde{\mathcal{M}}_g)_T$$

$= C$