

Thursday, November 13, 2014 11:23 AM

We know now that  $\mathcal{M}_g$  is an algebraic stack.

Why is it Deligne-Mumford

know from prior experience that Aut's of a curve of genus  $\geq 2$  are finite.

need  $\text{Aut}(C)$   $C/k=\bar{k}$   
 $\downarrow$   
 Spec  $k$   $\swarrow$  is unramified = étale.

really need to show that infinitesimal deformations of pts are unique when they exist

Compute deformations (§1.3.3) computation involves  $\mathcal{M}_g$ ?  
 $g \geq 2$  this set vanishes.

Now "Chap 9" Quasi-coherent sheaves on alg-stacks

Given an alg-stack  $\mathcal{X}/S$ , want to define a site for  $\mathcal{X}$

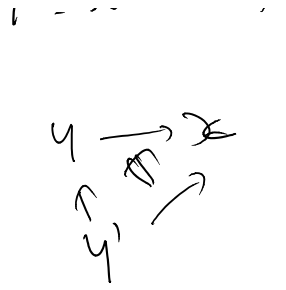
two approaches

$\mathcal{X}_+$   $\swarrow$  tag 06NT  
 underlying category =  $\mathcal{X} \xrightarrow{\pi} S$   
 morphisms cartesian arrows  
 $\&$  rows are maps  $X_i \rightarrow X$

$\searrow$   
 objects are morphisms of fibred cts  
 $Y \rightarrow \mathcal{X}$   
 $Y$  - scheme / space

$\tau$  = some topology on  $(Sch/S)$

s.t.  $\tau(x_i) \rightarrow \tau(x)$  are a cover in  $S_\tau$



2-yoneda:

$$\text{Hom}(Y, X) \xleftrightarrow{\text{equiv. of cats}} \mathcal{X}(Y)$$

$\uparrow$   
 fibred cat rep by a scheme  
 $(Sch/S)/Y$

Prop stack over a stack is a stack. (Tag 06N7)

i.e. if  $Y/C$  stack  $C$  site.  $\approx$  topology on  $Y$

if  $\mathcal{X}/Y$  is a stack then  $\mathcal{X}/C$  a fibred cat (by comp) of functors

and is a stack.

Prop  
(Tag 06N7)

If  $\mathcal{X} \xrightarrow{f} Y$  a morphism of cat. fibred in groupoids

$$\begin{array}{ccccc} & & \mathcal{X} & \xrightarrow{f} & Y \\ & \downarrow & \downarrow & & \downarrow \\ & & C & & \\ \exists \text{ a factorization} & & \mathcal{X} & \xrightarrow{a} & \mathcal{X}' & \xrightarrow{f'} & Y \\ & & \underbrace{\hspace{10em}} & & & & \\ & & & & & & f \end{array}$$

s.t.  $a$  is an equivalence

$f'$  is fibred in groupoids.

$\swarrow$  06N7

If  $\mathcal{X} \rightarrow Y$  is a morphism of stacks over  $C$   
 $\searrow C$   
 $\Rightarrow \mathcal{X}' \rightarrow Y$  (as above) is a stack!

Lisse-étale site of a stack  $\mathcal{X}$

$\mathcal{X}/S$  an algebraic stack,

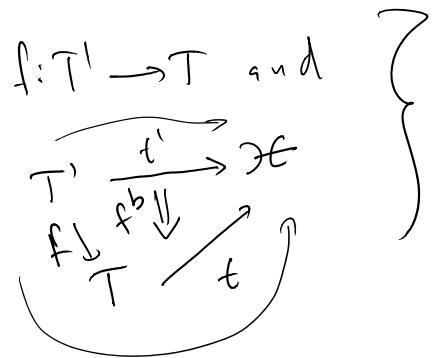
Def:  $\mathcal{X}$ -space is a pair  $(T, t)$   $T$  an alg. space  $/S$

$t: T \rightarrow \mathcal{X}$  a morphism of  $S$ -stacks.

$AlgSpec/\mathcal{X}$

want to make a category out of these

$$\text{Hom}((T', t'), (T, t)) = \{ (f, f^b) \mid f: T' \rightarrow T \text{ and } \dots \}$$



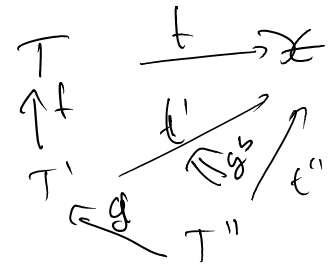
$$t', t \circ f \in \text{Hom}_{\text{fibcat}/S}(T', \mathcal{X})$$

a category

$$f^b: t' \xrightarrow{\sim} t \circ f$$

base preserving nat. trans.

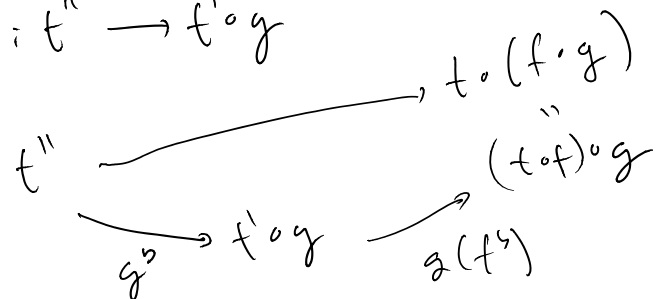
$$(T'', t'') \xrightarrow{(g, g^b)} (T', t') \xrightarrow{(f, f^b)} (T, t)$$



$$(f \circ g, g(f^b) \circ g^b)$$

$$f^b: t' \rightarrow t \circ f$$

$$g^b: t'' \rightarrow t' \circ g$$



ex:  $\mathcal{X}$  alg-space then this is the cat. of alg spaces/ $\mathcal{X}$

Def  $\text{Lis-Et}(\mathcal{X}) \subset \text{Sch}/\mathcal{X}$  pairs  $(T, t)$  s.t.  $T \xrightarrow{t} \mathcal{X}$   
smooth.  
and covers are families

$\{(t_i, t_i) \rightarrow (T, t)\}$  s.t.  $\{T_i \rightarrow T\}$  is an étale cov.