Lecture 26: Moduli of curves of general type is Deligne-Mumford. Stacks over stacks

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We "know" now that $\mathcal{M}_g$ is an algebraic stack.

Why is it Deligne-Mumford

Know from prior experience that Aut's of a curve of genus ≥ 2 are finite.

need $\underline{\mathrm{Aut}}(C)$ $C/_{k=\bar{k}}$

$\downarrow$ is unramified = étale.

Spec k

really need to show that infinitesimal deformations of auts are unique when they exist

Compute deformations (§1.3.3) computation involves Ω_{C}?

$g \geq 2$ this set vanishes.

Now "Chap 9" Quasi-coherent sheaves on alg. stacks

Given an alg. stack $\mathcal{X}/S$, want to define a site for $\mathcal{X}$

two approaches

tag 06NT

$\mathcal{X}_T$ underlying category $= \mathcal{X} \xrightarrow{\pi} S$

morphisms cartesian arrows

& covers are maps $X_i \to X$

objects are morphisms of fibred cats

$Y \longrightarrow \mathcal{X}$

Y - scheme/space

s.t. $\pi(x_i) \to \pi(x)$
are a cov in S_τ

τ = some topology
on (Sch/S)

$$\begin{tikzcd} \mathcal{Y} \arrow[r] & \mathcal{X} \\ \mathcal{Y}' \arrow[u] \arrow[ur] & \end{tikzcd}$$

2-yoneda: $\quad Hom(\mathcal{Y}, \mathcal{X}) \xleftrightarrow{\text{eq. of cats}} \mathcal{X}(\mathcal{Y})$

fibred
cat rep by a scheme
$(Sch/S)/\mathcal{Y}$

<u>Prop</u> stack over a stack is a stack. (Tag 06N7)

i.e. if $\mathcal{Y}/\mathcal{C}$ stack $\mathcal{C}$ a site. $\leadsto$ topology on $\mathcal{Y}$

& if $\mathcal{X}/\mathcal{Y}$ is a stack then $\mathcal{X}/\mathcal{C}$ a fibred cat (by comp of functors)

and is a stack.

<u>Prop</u> If
(Tag 06N7)

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

a morphism of cat. fibred in groupoids

$\exists$ a factorization $\quad \begin{tikzcd} \mathcal{X} \arrow[r, "a"] \arrow[rr, bend right, "f"'] & \mathcal{X}' \arrow[r, "f'"] & \mathcal{Y} \end{tikzcd}$

s.t. a is an equivalence
& f' is fibred in groupoids.

$\Downarrow$ 06N7

If $\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr] & \mathcal{Y} \arrow[d] \\ & \mathcal{C} \end{tikzcd}$ is a morphism of stacks over $\mathcal{C}$

$\Longrightarrow \mathcal{X}' \to \mathcal{Y}$ (as above)
is a stack!

Lisse-étale site of a stack $\mathcal{X}$

$\mathcal{X}/S$ an algebraic stack,

<u>Def</u>: $\mathcal{X}$-space is a pair (T,t) T an alg. space/S

$t: T \to \mathcal{X}$ a morphism of S-stacks. $\qquad$ $\mathrm{AlgSpc}/\mathcal{X}$

Want to make a category out of them

$$\mathrm{Hom}\big((T',t'),(T,t)\big) = \left\{ (f,f^b) \;\middle|\; f: T' \to T \text{ and } \begin{tikzcd} T' \arrow[r, "t'"] \arrow[d, "f"'] \arrow[dr, phantom, "f^b \Downarrow"] & \mathcal{X} \\ T \arrow[ur, "t"'] & \end{tikzcd} \right\}$$

$t', t\circ f \in \mathrm{HOM}_{\mathrm{fibcat}/S}(T',\mathcal{X})$

a category

$f^b : t' \xrightarrow{\sim} t\circ f$

base preserving nat. trans.

$$(T'',t'') \xrightarrow{(g,g^b)} (T',t') \xrightarrow{(f,f^b)} (T,t)$$

$$\big(f\circ g,\ g(f^b)\circ g^b\big)$$

$$\begin{tikzcd} T \arrow[r, "t"] & \mathcal{X} \\ T' \arrow[u, "f"] \arrow[ur, "t'"] \arrow[r, phantom, "\Uparrow g^b"] & \\ & T'' \arrow[ul, "g"] \arrow[uu, "t''"'] \end{tikzcd}$$

$f^b : t' \to t\circ f$

$g^b : t'' \to t'\circ g$

$$\begin{tikzcd} t'' \arrow[rr] \arrow[dr, "g^b"'] & & t\circ(f\circ g) = (t\circ f)\circ g \\ & t'\circ g \arrow[ur, "g(f^b)"'] & \end{tikzcd}$$

ex: $\mathcal{X}$ alg. space then this is the cat. of alg spaces/$\mathcal{X}$

Def $\text{Lis-Et}(\mathcal{X}) \subset Sch/\mathcal{X}$ pairs (T,t) s.t. $T \xrightarrow{t} \mathcal{X}$ smooth.

and covers are families

$$\{(T_i,t_i) \longrightarrow (T,t)\} \text{ s.t. } \{T_i \longrightarrow T\} \text{ is an étale cover.}$$