Lecture 27: Cohomology of quasi-coherent sheaves on algebraic stacks

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Let $\mathcal{X}/S$ an algebraic stack, F a sheaf on Lisse-étale site of $\mathcal{X}$. Then F is quasi-coherent if for all objects $T \xrightarrow{t} \mathcal{X}$ in Lisse-étale, t^*F is q. coherent on T

rest. of F to T_{et}.

Prop $F/\mathcal{X}$ q. coh $\Longleftrightarrow$ $\exists$ smooth surjection $X \xrightarrow{x} \mathcal{X}$ s.t. x^*F is q. coh on X.

Computing cohomology: 9.2

given a alg. stack $\mathcal{X}$ & smooth surjection $X \to \mathcal{X}$

$$X \times_{\mathcal{X}} X \rightrightarrows X \left(\to \mathcal{X}\right)$$

Simplicial objects: Δ = finite ordered sets, order preserving maps

ob $= \{[0,1,\dots,n]\}$

$\mathcal{C}$

$$\mathrm{Simp}(\mathcal{C}) = \mathrm{Fun}(\Delta^{op}, \mathcal{C})$$

$$\begin{matrix} \| \\ C_2 \; \substack{\to\\\to\\\to} \; C_1 \rightrightarrows C_0 \\ [0,1,2] \quad [0,1] \quad [0] \end{matrix}$$

Given any morphism $a \to b$ $\mathcal{C}$

$$\substack{\rightarrow\\\rightarrow\\\rightarrow} a \times_b a \times_b a \ \substack{\rightarrow\\\rightarrow\\\rightarrow} \ a \times_b a \rightrightarrows a$$

If $\mathcal{C}$ is a site, w/ final object e $\quad (Sch/S)$

can consider a cover $X \to e$

w/ simplicial object

$$X_\bullet = \left(\substack{\rightarrow\\\rightarrow\\\rightarrow} X \times_e X \rightrightarrows X \right)$$

$\mathcal{C}/X_\bullet$

General statement: If $\mathcal{C}$ a site, e final object

$X \xrightarrow{\pi} e$ cover. Then $H^i(\mathcal{C}, F) = H^i(\mathcal{C}/X_\bullet, \pi^{*\bullet} F)$

Ex: $\mathcal{X}$ alg. stack $X \xrightarrow{\pi} \mathcal{X}$ sm. surjection F q.coh.

then $H^i(\mathcal{X}, F) = H^i(X_\bullet, \pi^{*\bullet} F)$

even better: combinatorial way to compute cohom. at least

"simplicial sites"

Čech spectral seq: $H^p(X_q, F_q) \Rightarrow H^{p+q}(\mathcal{X}, F)$

$$\overset{\shortparallel}{E_1^{p,q}}$$

We translate to alg. spaces from stacks as above:

D... let $f: \mathcal{X} \to \mathcal{Y}$ q.compact & separated morphism of

Prop. ... alg stacks. $F/\mathcal{X}$ q. coherent. Then $R^i f_* F$ are q. coherent.

<u>Prop</u> if $X \longrightarrow \mathcal{X}$ smooth surjective, $\mathrm{Et}(X)/X_\bullet$

$X_\bullet = \left(\substack{\rightarrow\\ \rightarrow\\ \rightarrow} X \times_{\mathcal{X}} X \rightrightarrows X \right)$

there is an equiv. of cats $\quad QCoh\left(\mathrm{Et}(X)/X_\bullet \right) \simeq QCoh(\mathcal{X})$

(require each sheaf comprising objects to be q.coh.)