Lecture 28: Properties of stacks and morphisms between them

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Properties of stack & Morphisms between them

Constructions

Proper morphisms

$f: \mathcal{X} \to \mathcal{Y}$ is a morphism of alg-stacks represented by spaces. then usual def. of properness transfer from spaces.

Def $f: \mathcal{X} \to Y$ is closed if $\forall\ \mathcal{Z} \subset \mathcal{X}$ closed substacks. (w/ by closed embeddings)

$\mathcal{X}$: alg-stack, Y: alg. space

the image $\mathcal{Z}$ in Y is closed

(i.e. $Z \to \mathcal{Z} \xrightarrow{i} Y$ then we get $|Z| \to |Y|$

Z: alg space; $Z \to \mathcal{Z}$: sm. surj

top spaces.

want image of this to be closed)

Def $f: \mathcal{X} \to \mathcal{Y}$ morphism of alg. stacks is universally closed if $\forall\ Y \to \mathcal{Y}$ (Y alg. space), the pullback

$\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is closed.

Prop: If $f: \mathcal{X} \to \mathcal{Y}$ is representable, separated & finite type then f is univ. closed $\iff$ it is proper.

Def: $f:\mathcal{X}\to\mathcal{Y}$ is separated if $\Delta:\mathcal{X}\to\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}$ is proper.

Def: $f:\mathcal{X}\to\mathcal{Y}$ is proper if it is sep, f.type and univ. closed.

Prop: (10.1.4)

1) pullbacks of proper are proper
2) proper is local on the base w.r.t. to smooth top.
3) comp. of proper is proper
4) gf proper, g sep $\Rightarrow$ f proper
5) gf proper, f surj $\Rightarrow$ g proper.

EGA II 5.4.(2 or 3)

Relative Spec & Proj

Alg spaces: X/S $\mathcal{A}$ sheaf of algs on X

$$\underline{\mathrm{Spec}}_X(\mathcal{A})\,(T\to S) = \{ f: T\to X,\ f^*\mathcal{A}\to\mathcal{O}_T \}$$

fib. cat over X

$\underline{\mathrm{Spec}}_X(\mathcal{A})^{\text{fib cat}}$

$$\{T,\ T\xrightarrow{f}X,\ f^*\mathcal{A}\to\mathcal{O}_T\}$$

$\downarrow$

$$\{\mathrm{Hom}(f^*\mathcal{A},\mathcal{O}_T)\}$$
$$\downarrow$$
$$(T\to X)$$

$$\begin{tikzcd} \{T \to X\} \arrow[d] & \text{fib cat of } X \arrow[l] \\ T & Sch/S \arrow[l] \end{tikzcd}$$

q-coherent

$\mathcal{X}$ alg. stack, $\mathcal{A}$ sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras, (lisse-étale)

$\underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A})$ objects $\left(T,\ x \in \mathcal{X}(T),\ x^*\mathcal{A} \xrightarrow{\rho} \mathcal{O}_T \right)$

$\}$ gives $T \to \mathcal{X}$

morphism $(T', x', \rho') \longrightarrow (T, x, \rho)$

$T' \xrightarrow{g} T \qquad x' \xrightarrow{g^b} x$ over g

s.t.

$$\begin{tikzcd} x'^*\mathcal{A} \arrow[rr, "\text{“}g^b\text{''}"] \arrow[dr, "\rho'"'] & & g^*x^*\mathcal{A} \arrow[dl, "g^*\rho"] \\ & \mathcal{O}_T & \end{tikzcd}$$

← describe this canonical map

Remark: this is an étale stack since we have descent for q-coh sheaves.

Why is it algebraic

$$\begin{tikzcd} \underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A}) \arrow[r] & \mathcal{X} \\ \underline{\mathrm{Spec}}_T(f^*\mathcal{A}) \arrow[u] \arrow[r] & T \arrow[u, "f"'] \end{tikzcd}$$

is represented by schemes!

scheme

$$
\begin{array}{c}
| \\
\underline{\mathrm{Spec}}_T(f^*\mathcal{A}) \longrightarrow \text{scheme} \\
\uparrow \\
\text{scheme}
\end{array}
$$

"Scheme"/stack is a $\overset{\text{alg.}}{\wedge}$ stack

<u>Def:</u> $\pi : \mathcal{X} \to \mathcal{Y}$ is affine if $\forall\, \underset{\substack{\uparrow \\ \text{alg space}}}{Y} \to \mathcal{Y}$, $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is affine.

<u>Theorem</u> Have an equivalence of categories between

$\{$quasi-coherent sheaves of $\mathcal{O}_{\mathcal{Y}}$-algebras$\}$

$\leftrightarrow$ "$\{$Affine morphisms $\pi : \mathcal{X} \to \mathcal{Y}\}$"

<u>Proj:</u>

Given an alg-stack $\mathcal{X}$, and a q.coh. sheaf of graded $\mathcal{O}_{\mathcal{X}}$-algs

$\mathcal{A} = \bigoplus_{i \geq 0} \mathcal{A}_i$ $\qquad \underline{\mathrm{Proj}}_{\mathcal{X}}(\mathcal{A})$

via: $\underline{\mathrm{Proj}}_{\mathcal{X}}(\mathcal{A}) \longrightarrow \mathcal{X}$ fib. cat:

on an object $x \in \mathcal{X}(T)$ $\underline{\mathrm{Proj}}_{\mathcal{X}}(\mathcal{A})(x) = \mathrm{Proj}_T(x^*\mathcal{A})$

$x : T \longrightarrow \mathcal{X}$ ↗ sheaf w/ assoc. fib. cat / Sch/T.

i.e. objects of proj:

$$\begin{tikzcd} \underline{Proj}_{\mathcal{X}}(\mathcal{A}) \arrow[d] \\ Sch/S \end{tikzcd} = \left\{ x \in \mathcal{X},\ \begin{tikzcd} x : T \longrightarrow \mathcal{X} \arrow[d, mapsto] \\ T \end{tikzcd},\ T \longrightarrow Proj_T(x^* \mathcal{A}) \right\}$$

by def:

$$\begin{tikzcd} T \arrow[r] \arrow[drr, "="'] & Proj_T(x^*\mathcal{A}) \arrow[r] \arrow[d] & \underline{Proj}_{\mathcal{X}}(\mathcal{A}) \arrow[d] \\ & T \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

$\Longrightarrow$ Proj is rep. by schemes $\Rightarrow$ its an alg. stack.