

Properties of stack & Morphisms between them

Constructions

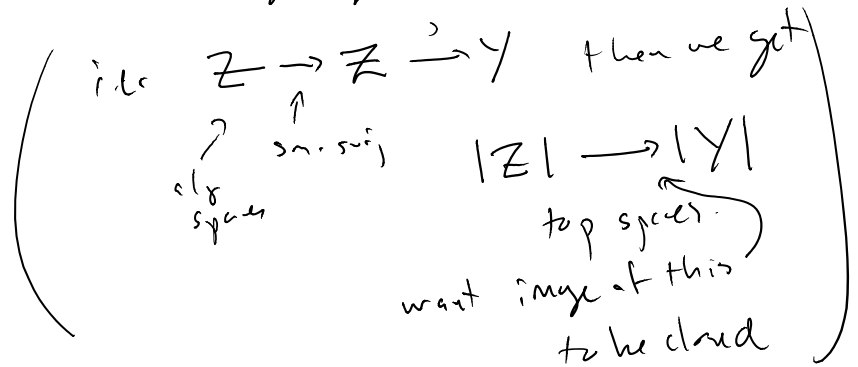
Proper morphisms

$f: X \rightarrow Y$ is a morphism of alg-stacks represented by spaces. then usual def. of properness transfer from spaces.

Def $f: X \rightarrow Y$ is closed if $\forall Z \subset X$ closed substacks. (w/ by closed embeddings)

$\begin{matrix} & & Y & \\ & \nearrow & & \nwarrow \\ \text{alg-stack} & & & & \text{alg-space} \end{matrix}$

the image Z in Y is closed



Def $f: X \rightarrow Y$ morphism of alg-stacks is universally closed if $\forall Y \rightarrow y$ the pullback

alg space,

$X \times_y Y \rightarrow Y$ is closed.

Prop: If $f: X \rightarrow Y$ is representable, separated & finite type then f is univ.-closed \Leftrightarrow it is proper.

Def: $f: X \rightarrow Y$ is separated if $\Delta: X \rightarrow X \times_Y X$ is proper.

Def: $f: X \rightarrow Y$ is proper if it is sep, f-type, and univ-closed.

Prop: (10.1.4)

1) pullbacks of proper are proper

2) proper is local on the base w/r to smooth top.

3) comp. of proper is proper

4) $g \circ f$ proper, g sep $\Rightarrow f$ proper

5) $g \circ f$ proper, f surj $\Rightarrow g$ proper.

EGA II 5.4. (2-3)

Relative Spec & Proj

Alg spaces: X/S A sheaf of algs on X

$$\text{Spec}_X(\mathcal{A}) (T \rightarrow S) = \{ f: T \rightarrow X, f^* \mathcal{A} \rightarrow \mathcal{O}_T \}$$

fibrat on X

$$\{ \text{Hom}(f^* \mathcal{A}, \mathcal{O}_T) \}$$

\downarrow

$$(T \rightarrow X)$$

$$\text{Spec}_X(\mathcal{A}) \text{ fibrat}$$

$$\{ T, T \xrightarrow{f} X, f^* \mathcal{A} \rightarrow \mathcal{O}_T \}$$

\downarrow

↓
 $\{T \rightarrow X\} \leftarrow \text{fib cat at } X$

↓
 $T \leftarrow \text{Schemes}$
 $\quad \quad \quad \uparrow$
 $\quad \quad \quad \text{q-coherent}$

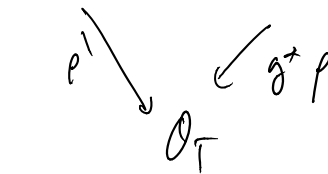
\mathcal{X} alg. stack, \mathcal{A} sheaf of $\mathcal{O}_{\mathcal{X}}$ -algebras, (lisse-étale)

$\text{Spec}_{\mathcal{X}}(\mathcal{A})$ objects $(T, x \in \mathcal{X}(T), x^* \mathcal{A} \rightarrow \mathcal{O}_T)$
 $\quad \quad \quad \downarrow$
 $\quad \quad \quad T \rightarrow \mathcal{X}$

morphism $(T', x', \rho) \rightarrow (T, x, \rho)$

$T' \xrightarrow{g} T \quad x' \xrightarrow{g^b} x$ or g

s.t. $x'^* \mathcal{A} \xrightarrow{\delta} g^* x^* \mathcal{A}$ ← describe this canonical map



Remark: this is an étale stack since we have descent for q-coh schemes.

Why is it algebraic

$\text{Spec}_{\mathcal{X}}(\mathcal{A}) \rightarrow \mathcal{X}$ is represented by schemes!
 $\quad \quad \quad \uparrow \quad \quad \quad \uparrow$
 $\text{Spec}_T(\mathcal{A}) \rightarrow T$
 $\quad \quad \quad \text{scheme}$

$$\begin{array}{ccc} & \uparrow & \\ \text{Spec } T(\mathbb{A}^1) & \longrightarrow & \text{scheme} \\ & \uparrow & \\ & \text{scheme} & \end{array}$$

"Scheme/stack is a stack
 \uparrow
 alg"

Def $\pi: X \rightarrow Y$ is affine if $\forall Y \rightarrow Y$, $X \times_Y Y \rightarrow Y$ is affine.
 \uparrow
 abs spec

Theorem Have an equivalence of categories between
 $\{ \text{quasi-coherent sheaves of } \mathcal{O}_Y\text{-algebras} \}$

\uparrow " $\{ \text{Affine morphisms } \pi: X \rightarrow Y \}$ "

Prop

Given an alg-stack X , and a quoh. sheaf of graded \mathcal{O}_X -algs

$$\mathcal{A} = \bigoplus_{i \geq 0} \mathcal{A}_i \quad \text{Proj}_X(\mathcal{A})$$

via $\text{Proj}_X(\mathcal{A}) \rightarrow X$ fib. cat:

over an object $x \in X$ $\text{Proj}_X(\mathcal{A})(x) = \text{Proj}_T(x^* \mathcal{A})$

$$x: T \rightarrow \mathcal{X}$$

↑
 stack w/ assoc.
 fib. cat / Sch/T.

i.e. objects of Proj:

$$\text{Proj}_{\mathcal{X}}(A) = \{ x \in \mathcal{X}, x: T \rightarrow \mathcal{X}, T \rightarrow \text{Proj}_T(x^*A) \}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \text{Sch/S} & & T \end{array}$$

$$\text{by def: } \begin{array}{ccccc} T & \longrightarrow & \text{Proj}_T(x^*A) & \longrightarrow & \text{Proj}_{\mathcal{X}}(A) \\ & \searrow & \downarrow & & \downarrow \\ & & T & \xrightarrow{x} & \mathcal{X} \end{array}$$

⇒ Proj is rep. by schemes. ⇒ its an alg. stack.