

Today:

- Root stacks ✓
- Coarse Moduli ✓
- Gerbes

Root Stacks

$X, L$  line bundle (or close to (center divisor))

want to construct:  $X(L, 1/n) \xrightarrow[\text{stack}]{\pi} X$  s.t. on  $X(L, 1/n) \exists \mathcal{L}$

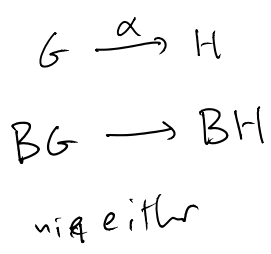
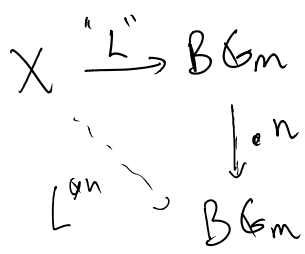
s.t.  $\mathcal{L}^{\otimes n} = \pi^* L,$

$X(L, 1/n)$  universal for this.

Idea: Line bundles are in bijection w/  $G_m$ -torsors

$$B G_m = [\bullet / G_m]$$

$$L \in B G_m(X) \iff L: X \rightarrow B G_m \text{ \small 2-Yoneda}$$



$(g_{ij} \in G(U_{ij})) \rightsquigarrow \alpha(g_{ij}) \in H(U_{ij})$   
 transition

Right  $G$ -torsor  $\mathbb{A}^1/G$

$$g: (f, h) \rightarrow (fg^{-1}, \alpha(g)h)$$

right  $H$ -action

How to make  $X(L, 1/n)$ ?

$$\begin{array}{ccc} X(L, 1/n) & \xrightarrow{\mathcal{L}} & B\mathbb{G}_m \\ \downarrow \Gamma & & \downarrow \eta \\ X & \xrightarrow{h} & B\mathbb{G}_m \end{array}$$

not hard to see:

$X(L, 1/n)$  is a DM stack  
if  $n$  not divisible by  
char  $S$   
( $1/n \in \mathcal{O}(S)$ )  
 $S = \bullet$  here.

Similar for Cartier-Like

$[A^1/G_m]$  similar on map.

### Coarse Moduli

Def  $\mathcal{X}/S$  an alg. stack. A coarse moduli space for  $\mathcal{X}/S$   
is an algebraic space  $X/S$  together w/ a morphism  
 $\pi: \mathcal{X} \rightarrow X$  s.t.

(1) (Universal)  $\forall Y/S$  alg-spaces  $\mathcal{X} \rightarrow Y$ , can factor uniquely

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & Y \\ \pi \searrow & & \nearrow \\ & X & \end{array}$$

(2)  $\mathbb{A}^1$  algebraic closed field,  $\mathcal{X}(k) / \text{iso} \xrightarrow{\pi} X(k)$  is bijective

Thm (Keel-Mori) / (Conrad) If  $S$  loc. noeth, and  $\mathcal{X}$  loc. finitely presented /  $S$

w/ finite diagonal, then  $\exists$  coarse moduli space. Further

1)  $X/S$  loc. finite type and if  $\mathcal{X}/S$  sep  $\Rightarrow X/S$  sep

2)  $\pi: \mathcal{X} \rightarrow X$  is proper and  $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{\mathcal{X}}$  is an iso.

3) If  $X' \rightarrow X$  flat morphism of alg. spaces, then

$\pi^{-1}: \mathcal{X} \times_X X' \rightarrow X'$  is a coarse moduli.

Thm "Orbitoid"

Let  $\mathcal{X}/S$  sep. DM stack,  $S$  loc. Noeth,  $\mathcal{X} \xrightarrow{\pi} X$  coarse moduli,  $\bar{x} \rightarrow \mathcal{X}$ , geom pt,  $\bar{x} \rightarrow X$  image in  $X$

$G_{\bar{x}} = \text{Aut}(\bar{x})$  f. gp scheme / res field  $k_{\bar{x}} = \text{alg. closed}$

" finite group

then  $\exists \bar{x} \in U \subset X$  open  $\exists$   $v \rightarrow u$  finite w/  $G_{\bar{x}}$  action

sol.  $\mathcal{X} \leftarrow [v/G_{\bar{x}}] \leftarrow \bar{x}$

$\downarrow \quad \downarrow$   
 $X \leftarrow u \leftarrow \bar{x}$

Prop If  $\mathcal{X}/S$  is a DM stack of finite type over loc. Math  $S$   
 $\pi: \mathcal{X} \rightarrow X$  coarse moduli. Suppose that  $\mathcal{X}$  is tame  
 then the functor  $\pi_*$  is exact on q.coh. sheaves.

Consequently if  $\mathcal{F}$  is q.coh. on  $\mathcal{X}$ , then  $(R^n \pi_*) \mathcal{F} = (R^n \pi_*) (\pi_* \mathcal{F})$   
 $\parallel$   $\parallel$   
 $H^n(\mathcal{X}, \mathcal{F})$   $H^n(X, \pi_* \mathcal{F})$   
 if  $S$  affine  $\rightarrow$  if  $S$  affine



tame =  $\mathcal{H} \xrightarrow{\sim} \mathcal{X} \hookrightarrow \mathcal{X}$  geom pt,  $|\text{Aut}(\mathcal{X})| \in k^*$   
 $\parallel$   
 $\text{Spec } k$

Pf: if taking quot by grp actions:

$$V = \text{Spec } S \curvearrowright G \quad \pi_* M = M^G$$

$$\begin{array}{ccc} m & \longrightarrow & n \\ M & \twoheadrightarrow & N \end{array}$$

$$\begin{array}{ccc} M^G & \twoheadrightarrow & N^G \\ ? & & ? \\ \vdots & & \vdots \\ \longrightarrow & & \longrightarrow \end{array}$$

$$\frac{1}{|G|} \sum_{g \in G} g(m)$$

## Gerbes

$\mu$ -comm. gp scheme /  $S$  (affine)

$$R_{\mu}(X) \simeq H^1(X, \mu)$$

$\mu$ -Torsors

$B_\mu$

$$B_\mu(X) / \text{iso} \stackrel{b_{ij}}{\simeq} H^1(X, \mu)$$

Čech cohom.

w/  $\mathcal{M}$  action  $\rightarrow$   
 $\mathcal{G}$   
 $\downarrow$   
 $X$

s.t.  $\exists$  cover  $X_i \rightarrow X$

$\downarrow$  isoms  $\mathcal{G}|_{X_i} \simeq \mathcal{M}|_{X_i}$

$$\begin{array}{ccc}
 \mathcal{G}|_{X_i} & \xrightarrow{\alpha_i|_{X_{ij}}} & \mathcal{M}|_{X_i}|_{X_{ij}} \\
 \downarrow & \searrow \alpha_j^{-1} \alpha_i = \varphi_{ij} & \parallel \\
 \mathcal{G}|_{X_j} & \xrightarrow{\alpha_j} & \mathcal{M}|_{X_j}|_{X_{ij}}
 \end{array}$$

$$\varphi_{ij} \in \mathcal{M}(X_{ij})$$

$\varphi_{jk} \varphi_{ij} = \varphi_{ik} \rightarrow$  Čech  
cocycle

eq. class doesn't care about choice  
of isos  $X_i$  (cohomology)

$$\underline{\text{ex: line bundles}} / \text{iso} = H^1(X, \mathcal{O}_X)$$

$H^2(X, \mu)$ : gerbes. (Giraud's thesis)

Informal def:  $\mu$ -gerbe/ $X$  is a stack on  $X$  locally  $\simeq$  to  $B_\mu/X$

Def:  $\mu$ -gerbe on  $\mathcal{C}$  is a stack  $\mathcal{G} \xrightarrow{P} \mathcal{C}$  together w/ an

action of  $\mu$  on all objects (compatible)

i.e.  $\forall X \in \mathcal{D}$

$$\text{Aut}(X) \leftarrow \mu|_{(C/X)}$$

? sheaf on  $(C/X)$

s.t. 1)  $\forall X \in C \exists \text{cong } \{X_i \rightarrow X\}$  s.t.  $\mathcal{D}(X_i) \neq \emptyset$  all  $i$

2)  $\forall Y_1, Y_2 \in \mathcal{D}$  w/  $pY_1 = pY_2 \exists \text{ cov } \{X_i \rightarrow X\}$

s.t.  $Y_1|_{X_i} \cong Y_2|_{X_i}$  all  $i$

3)  $\mu$  acts on all compatible.

connection to  $H^2(X, \mu)$

$$\alpha_{ijk} \in \mu(X_{ijk}) \quad \alpha_{ijt} \alpha_{ije}^{-1} \alpha_{ike} \alpha_{jke}^{-1} = 1$$

char groups  $G \sim \mu \rightarrow G$  centrally

cat of twisted  $G$ -torsors

normal:  $g_{ij} \in X_{ij}$

$$g_{jk} g_{ij} = g_{ik}$$

$\alpha$ -twisted:  $g_{ij}$  s.t.  $g_{jk} g_{ij} = \alpha_{ijk} g_{ik}$

$G$ -tors on  $X_i$  trivial

gives data  $h_{ij} g_{ij}$

maps are locally maps of  $G$ -torsors compat w/  $g_{ij}$

local action is  $\mu$ 's  $\mu \rightarrow G$ .

have  $\alpha_{ijl} = 0$

locally  $\alpha_{ijl} = 0$

$$G \rightarrow \mu \rightarrow G \rightarrow G/\mu \rightarrow 0$$

$$H^1 \quad H^1 \rightarrow H^2(\mu)$$

~~the~~

$\alpha$ -in  $G$  knows  
"

$G/\mu$  knows w/ local lifts,  
fairly to give.