Lecture 29: root stacks, coarse moduli and gerbes

Tuesday, December 2, 2014   11:17 AM

Today:

- Root stacks ✓
- Coarse Moduli ✓
- Gerbes

Root Stacks

X, L line bundle (or close to (Cartier divisor))

want to construct: $X(L, 1/n) \xrightarrow[\text{stack}]{\pi} X$ s.t. on $X(L, 1/n)$ $\exists\, \mathcal{L}$

s.t. $\mathcal{L}^{\otimes n} = \pi^* L$,

$X(L, 1/n)$ universal for this.

Idea: Line bundles are in bijection w/ $\mathbb{G}_m$-torsors

$B\mathbb{G}_m = [\circ / \mathbb{G}_m]$ $L \in B\mathbb{G}_m(X) \longleftrightarrow L : X \to B\mathbb{G}_m$
2-Yoneda

$$\begin{tikzcd} X \arrow[r, "``L"'"] \arrow[dr, dashed, "L^{\otimes n}"'] & B\mathbb{G}_m \arrow[d, "\otimes n"] \\ & B\mathbb{G}_m \end{tikzcd}$$

$G \xrightarrow{\alpha} H$

$BG \longrightarrow BH$

via either

$\varphi_{ij} \in G(U_{ij}) \rightsquigarrow \alpha(\varphi_{ij}) \in H(U_{ij})$
transition

Right G-torsor $/G$

$g : (f,h) \to (fg^{-1}, a(g)h)$

right H-action

How to make $X(L, 1/n)$?

$$\begin{tikzcd} X(L,1/n) \arrow[r, "\mathcal{L}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B\mathbb{G}_m \arrow[d, "n"] \\ X \arrow[r, "L"] & B\mathbb{G}_m \end{tikzcd}$$

not hard to see: $X(L, 1/n)$ is a DM stack if n not divisible by char's ($1/n \in \mathcal{O}(S)$), S = base.

Similar for Cartier-Like $[\mathbb{A}^1/\mathbb{G}_m]$ similar on map.

<u>Coarse Moduli</u>

<u>Def</u> $\mathcal{X}/S$ an alg. stack. A <u>coarse moduli space</u> for $\mathcal{X}/S$ is an algebraic space. X/S together w/ a morphism $\pi : \mathcal{X} \to X$ s.t.

(1) (Universal) $\forall Y/_S$ alg-spaces $\mathcal{X} \to Y$, can factor uniquely

$$\begin{tikzcd} \mathcal{X} \arrow[rr] \arrow[dr, "\pi"'] & & Y \\ & X \arrow[ur] & \end{tikzcd}$$

(2) $\forall$ alg. closed field k, $\mathcal{X}(k)/_{iso} \xrightarrow{\pi} X(k)$ is bijective

<u>Thm</u> (Keel-Mori /Conrad) If S loc. noeth, and $\mathcal{X}$ loc. finitely presented/S w/ <u>finite</u> diagonal, then $\exists$ coarse moduli space. Further

1) X/S loc. finite type and if $\mathcal{X}/S$ sep $\Rightarrow X/S$ sep

2) $\pi : \mathcal{X} \to X$ is proper and $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an iso.

3) If $X' \to X$ flat morphism of alg. spaces, then $\pi' : \mathcal{X} \times_X X' \to X'$ is a coarse moduli.

<u>Thm</u> "Orbifold"

Let $\mathcal{X}/S$ sep. DM stack, S loc. Noeth, $\mathcal{X} \xrightarrow{\pi} X$ coarse moduli, $\tilde{x} \to \mathcal{X}$, geom pt, $\bar{x} \to X$ image in X

$G_{\bar{x}} = \mathrm{aut}(\tilde{x})$ f. gp scheme / res field of $\tilde{x}$ = alg closed

" finite group

then $\exists \bar{x} \in U \subset X$ open & $V \to U$ finite w/ $G_{\bar{x}}$ action

s.t.
$$\begin{array}{ccccc} \mathcal{X} & \leftarrow & [V/G_{\bar{x}}] & \hookleftarrow & \tilde{x} \\ \downarrow & & \downarrow & & \\ X & \hookleftarrow & U & \hookleftarrow & \bar{x} \end{array}$$

Prop If $\mathcal{X}/S$ is a DM stack of finite type over loc. Noeth S
$\pi : \mathcal{X} \to X$ coarse moduli. Suppose that $\mathcal{X}$ is tame
then the functor π_* is exact on q.coh. sheaves.

Consequently if $\mathcal{F}$ is q.coh. on $\mathcal{X}$, then
$$\underbrace{(R^n p_*)\mathcal{F}}_{\substack{H^n(\mathcal{X},\mathcal{F}) \\ \text{if } S = \text{affine}}} = \underbrace{(R^n p_*)(\pi_* \mathcal{F})}_{\substack{H^n(X, \pi_* \mathcal{F}) \\ \text{if } S = \text{affine}}}$$

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \arrow[dr, "\pi"] & \\ S & X \arrow[l, "p"] \end{tikzcd}$$

tame $= \forall\ \underset{\substack{\| \\ \operatorname{Spec} k}}{\tilde{x}} \hookrightarrow \mathcal{X}$ geom pt, $|\mathrm{Aut}(\tilde{x})| \in k^*$

Pf: if taking quots by gp actions:

$$V = \operatorname{Spec} S \circlearrowleft G \qquad \pi_* M = M^G$$

$$m \longmapsto n$$
$$M \twoheadrightarrow N$$
$$M^G \overset{?}{\twoheadrightarrow} N^G \ni n$$
$$\frac{1}{|G|} \sum_{g \in G} g(m) \longmapsto$$

Gerbes

μ - comm. gp scheme$/S$ (affine)

$$R_\mu(X)/ \overset{\text{bij}}{\simeq} H^1(X, \mu)$$

μ-Torsors

$B\mu$

$$B\mu(X)/_{\text{iso}} \overset{\text{bij}}{\simeq} H^1(X,\mu)$$

Čech cohom.

$$\begin{array}{c} \mathcal{F} \\ \downarrow \\ X \end{array}$$

w/ μ action

s.t. $\exists$ cover $X_i \to X$ & isoms $\mathcal{F}|_{X_i} \simeq \mu|_{X_i}$

$$\begin{tikzcd} \mathcal{F}|_{X_i}|_{X_{ij}} \arrow[r, "\alpha_i|_{X_{ij}}", "\sim"'] \arrow[d, bend left] & \mu|_{X_i}|_{X_{ij}} \arrow[d, equal] \\ & \alpha_j^{-1}\alpha_i = \varphi_{ij}\ \mu|_{X_i \cap X_j} \arrow[d, equal] \\ \mathcal{F}|_{X_j}|_{X_{ij}} \arrow[r, "\sim", "\alpha_j"'] & \mu|_{X_j}|_{X_{ij}} \end{tikzcd}$$

$\varphi_{ij} \in \mu(X_{ij})$

$\varphi_{jk}\varphi_{ij} = \varphi_{ik} \to$ Čech cocycle

eq. class doesn't care about choice of isos α_i (cohomologous)

ex: line bundles$/_{\text{iso}} = H^1(X, \mathbb{G}_m)$

$H^2(X,\mu)$: gerbes. (Giraud's thesis)

Informal def: μ-gerbe$/X$ is a stack on X locally $\simeq$ to $B\mu/X$

Def μ-gerbe on $\overset{\text{site}}{\mathcal{C}}$ is a stack $\mathcal{S} \xrightarrow{p} \mathcal{C}$ together w/ an action of μ on all objects (compatible)

i.e. $\forall X \in \mathcal{D}$ $\qquad \underline{Aut(X)} \leftarrow \mu|_{(C/X)}$

sheaf on (C/X)

s.t. 1) $\forall X \in C$ $\exists$ cov $\{X_i \to X\}$ s.t. $\mathcal{D}(X_i) \neq \emptyset$ all i

2) $\forall Y_1, Y_2 \in \mathcal{D}$ w/ $pY_1 = pY_2$ $\exists$ cov $\{X_i \to X\}$

s.t. $Y_1|_{X_i} \simeq Y_2|_{X_i}$ all i

3) μ actions all compatible.

connection to $H^2(X, \mu)$

$\alpha_{ijk} \in \mu(X_{ijk})$ $\qquad \alpha_{ijk} \alpha_{ijl}^{-1} \alpha_{ikl} \alpha_{jkl}^{-1} = 1$

choose gpsch G w $\mu \hookrightarrow G$ centrally

cat of α-twisted G ~~torsors~~

normal: $g_{ij} \in X_{ij}$ $\qquad g_{jk} g_{ij} = g_{ik}$

α-twisted: g_{ij} s.t. $g_{jk} g_{ij} = \alpha_{ijk} g_{ik}$

G-tors on X_i trivial $\qquad$ maps are locally maps of

glu data X_{ij} g_{ij} $\qquad$ G-torsors comp w/ g_{ij}

... local action via μ's $\mu \hookrightarrow G$.

have [illegible]

locally $\alpha_{ijk} = 0$

$0 \to \mu \to G \to G/\mu \to 0$

$H^1 \quad H^1 \to H^2(\mu)$

α-ln G torsors

G/μ torsor w/ local lifts, failing to glue