Lecture 3: Schemes and Zariski gluing of sheaves

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- Schemes as functors
- Grothendieck topologies, sites, sheaves and topoi
- Big, small, medium and tiny sites: Zariski, etale, fppf, lisse-etale, smooth

Organizational stuff: course page, issue tracker (emails), distribution of Olsson's textbook

__Basic Idea:__ Yoneda Lemma

$\mathcal{C}$ - category

There's a natural full & faithful embedding of $\mathcal{C}$ into the category

$\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$

$$X \in \mathrm{Ob}(\mathcal{C}) \rightsquigarrow h_X : \mathcal{C} \longrightarrow \mathrm{Sets}$$
$$Y \longmapsto \mathrm{Hom}(Y, X)$$

we'll consider functors from $\mathcal{C}$ = Affine schemes or schemes or ..

__Useful construction:__ fiber product of functors

$H, G, F : \mathcal{C} \longrightarrow \mathrm{Sets}$

$$\begin{tikzcd} & F \arrow[dr] & \\ F \times_H G \arrow[ur] \arrow[dr] & & H \\ & G \arrow[ur] & \end{tikzcd}$$

defined so that $\forall X \in \mathrm{Ob}(\mathcal{C})$

$$F \times_H G(X) = F(X) \times_{H(X)} G(X)$$

Nice thing about fiber prods,

$$\begin{tikzcd} F \arrow[r] & G \\ h_Y \arrow[u, dashed] \arrow[r, dashed] & h_X \arrow[u] \\ Y \arrow[r] & X \end{tikzcd}$$

<u>Def</u>: $\varphi: F \to G$ relatively representable if $\forall X \in \mathcal{C}$, $g: h_X \to G$, the fiber product $F \times_G h_X$ is representable.

In particular if $\mathcal{C}$ = Schemes, can talk about (formally) smooth, étale, unramified, morphisms q.compact, affine.

Affine Schemes $\hookrightarrow$ Schemes (separated)

$$\begin{array}{c} \underline{Aff} \hookrightarrow \mathrm{Func}(\underline{Aff}, \underline{Sets}) \\ \cup \\ \text{Schemes} \\ \cup \\ \text{Sep. schemes} \\ \cup \\ \text{Alg. Spaces} \end{array}$$

all functors in the following are here

<u>Def</u> $f: F \to G$ is an affine open (closed) embedding if

i) f rel. representable

ii) $\forall X \in \underline{Aff}$ $g: h_X \to G$ then $F \times_G h_X \to h_X$ "is" an open (closed) embedding.

Def: F is a big Zariski sheaf if $\forall U \in \mathrm{Ob}(\underline{Aff})$

U_i cover of U

$$F(U) \to \prod F(U_i) \rightrightarrows \prod F(U_{ij}) \qquad U_{ij} = U_i \cap U_j$$

is an equalizer

$$\alpha_i \in F(U_i) \quad \alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$$

<u>Prop/Def</u> A functor F is representable by a (separated) scheme if and only if

i) F is a big Zariski sheaf

(ii) $\Delta : F \to F \times F$ is an affine closed embedding)

iii) $\exists$ a family $\{X_i\}$ in <u>Aff</u> & morphisms $h_{X_i} \to F$ which are affine open embeddings such that $\coprod_i h_{X_i} \to F$ is <u>surjective</u>

<u>Rem</u>: $F \xrightarrow{\varphi} G$ big Zariski sheaves is surjective if $\forall\ U \in$ <u>Aff</u>, $u \in G(U)$ $\exists \{U_i\}$ cover of U, $v_i \in F(U_i)$ s.t. $\varphi(U_i) v_i = u|_{U_i}$

<u>Grothendieck topologies</u>

Basic problem: Zariski is inadequate.

étale topology

<u>Idea</u>: enlarge notion of a cover

Rough terms: étale cover is a combination of

- Zariski open sets
- finite covering spaces

<u>Def</u> $\mathcal{C}$ - category. A Grothendieck topology on $\mathcal{C}$ consists of a set $\mathrm{Cov}(X)$ of collections $\{X_i \to X\}_{i \in I}$ of morphisms in $\mathcal{C}$ for every $X \in \mathrm{Ob}(\mathcal{C})$ s.t.

i) if $V \to X$ is an iso. then $\{V \to X\} \in \mathrm{Cov}(X)$

ii) $\{X_i \to X\} \in \mathrm{Cov}(X)$, and $Y \to X$ a morph then

$$\{X_i \times_X Y \to Y\} \in \mathrm{Cov}(Y)$$

iii) $\{X_i \to X\} \in \mathrm{Cov}(X)$, $\{V_{ij} \to X_i\} \in \mathrm{Cov}(X_i)$

$$\Rightarrow \{V_{ij} \to X\} \in \mathrm{Cov}(X)$$

$$\begin{tikzcd} V_{ij} \arrow[rr] \arrow[dr, dotted] & & X \\ & X_i \arrow[ur, dotted] & \end{tikzcd}$$

<u>Def</u> A Site = a category together w/ a Groth. top.