Lecture 5: topoi, sites and points

Tuesday, September 2, 2014 10:59 AM

Recall: A topos T is a category which is equivalent to the category of sheaves on some site $\mathcal{C}$.

Reccomend: look at Exercise 2L

Def: If $\mathcal{C}, \mathcal{C}'$ are sites, a continuous map $f: \mathcal{C}' \to \mathcal{C}$ is (f^{-1}) a functor $f: \mathcal{C}' \to \mathcal{C}$ s.t. $\forall X \in \mathcal{C}'$, $\{X_i \to X\} \in \mathrm{Cov}(X)$ the family $\{f(X_i) \to f(X)\} \in \mathrm{Cov}(f(X))$ and f commutes w/ fiber products when they exist in $\mathcal{C}'$.

Needed (sites)$^{\wedge}$: $\{X_i \to X\}$ cov, $Y \to X$ any morphism

$\{X_i \times_X Y \to Y\}$ cover. (and fiber products exist!)

Def: If T & T' are topoi, a morphism $f: T \to T'$ is an adjoint pair of functors

$f_*: T \to T'$ ← right

$f^*: T' \to T$ ← left

& naturalisom $\mathrm{Hom}_T(f^* -, -) \cong \mathrm{Hom}_{T'}(-, f_* -)$ of bifunctors.

such that f^* commutes w/ finite limits.

Def: If $f: \mathcal{C}' \to \mathcal{C}$ is cont. map of sites, T, T' assoc. topoi

$\mathrm{Shv}(\mathcal{C})$ " $\mathrm{Shv}(\mathcal{C}')$ "

then can define $f_*: T \to T'$

via $(f_*F)(U) = F(f(U))$

$U \in \mathcal{C}'$

this is a sheaf.

$$\begin{array}{ccccc}
(f_*F)(U) & \longrightarrow & \prod_i (f_*F)(U_i) & \rightrightarrows & \prod_{i,j} (f_*F)(U_i \times_U U_j) \\
\| & & \| & & \| \\
F(f(U)) & \longrightarrow & \prod F(f(U_i)) & \rightrightarrows & \prod F(f(U_i \times_U U_j)) \\
& & & & \| \\
& & & & F(f(U_i) \times_{f(U)} f(U_j))
\end{array}$$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{exact since } F \text{ a sheaf}}$

<u>Rem</u>: f_* as defined above also gives a functor

$f_* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}'$ of presheaf cats.

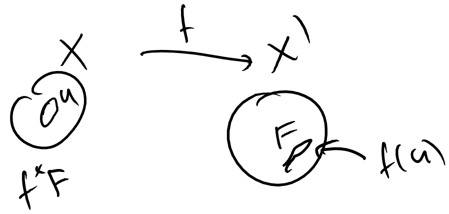
<u>Prop</u>: if $f : \mathcal{C}' \longrightarrow \mathcal{C}$ cont. morph. of sites then

$f_* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}'$ has a left adjoint $\hat{f}^*$

(then get a left adjoint on sheaves via $f^*(F) = (\hat{f}^* F)^a$

<u>Pf. sketch</u>:

Idea



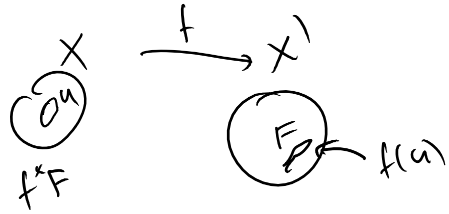
$$f^*F(U) = \varinjlim_{f(U) \subset V} F(V)$$

$\|$

$V,\ U \subset f^{-1}(V)$

Pf. sketch:

Idea



$$f^*F(U) \neq \varinjlim_{f(U)\subset V} F(V)$$

$$\Vert$$

$$V,\ U \subset f^{-1}(V)$$

$$f^*F(U) = \varinjlim_{\text{"}V\text{'s}\ U \to f(V)\text{"}} F(V)$$

given U, define

I_U: Cat w/ objects

(V, ρ) $\quad \rho : U \to f(V)$

$$f^*F(U) = \varinjlim_{(V,\rho)\in I_U} F(V)$$

$(V,\rho) \to (W, \varepsilon)$

$V \to W$ s.t.

$$\begin{tikzcd} & U \arrow[dl, "\rho"'] \arrow[dr, "\varepsilon"] & \\ f(V) \arrow[rr] & & f(W) \end{tikzcd}$$

commutes.

Q: If $C' \xrightarrow{f} C$ continuous, is it true that

$f^*(h_U) = h_{f(U)}$ (or when is this true?)

limit $= \varprojlim$

colim $= \varinjlim$

Prop If $C' \xrightarrow{f} C$ is continuous and if

- C' has all finite limits
- f commutes w/ finite limits

then $f^* : T' \to T$ commutes w/ finite limits & hence

$f = (f_*, f^*, \ )$ is a morphism of topoi.

Cohomology Exists

T topos, Λ a ring object in T $\quad (T, \Lambda)$ ringed topos.

Mod_Λ = cat of Λ-modules (sheaves of Λ-mods)

Thm Mod_Λ has enough injectives. (Stacks Project Tag 01DQ)

Usual proof: $\quad \mathcal{F} \hookrightarrow \prod_{x \in X} i_{x*}(\underbrace{i_x^* \mathcal{F}}_{\text{stalk at } x})$

$i_x^* \mathcal{F}$ is a $i_x^* \Lambda$-mod, can stick into an injective one I_x

$$\mathcal{F} \hookrightarrow \prod_{x \in X} i_{x*} i_x^* \mathcal{F} \hookrightarrow \prod_{x \in X} (i_x)_*(I_x)$$

Def $pt = Shv(pt) \cong Sets$.

$$\begin{array}{ccc} (S, \{a\}) & \hookrightarrow & (S, *) \\ \downarrow & & \\ (T, \{b\}) & & \end{array}$$

Def a point in a topos T is a morphism of topoi

$pt \to T \qquad\qquad f^*: T \to Sets$

Def T has enough points if $\exists$ a set X of points s.t.

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$$T \longrightarrow Set^X$$
$$F \longmapsto \{x^*F\}_{x \in X} \text{ is faithful.}$$

Ex: X top space, $T = Shv(X)$ then if $x \in X$

$\{x\} \longrightarrow X$ top space map gives a morphism of sites & map of topoi & a point $x \in T$ $\quad x : pt \to T$

x^*, x_*

$x^* F = F_x$

X are enough pts for T

Given enough pts, do $\square$

$$\begin{tikzcd} & N \\ I \arrow[r, hook] \arrow[ur] & M \arrow[u, dotted] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{G} \\ \mathcal{I} \arrow[r, hook] \arrow[ur] & F \arrow[u, dotted] \end{tikzcd}$$

$\mathcal{I} = i_* I$