Lecture 6: Fibered categories

Thursday, September 4, 2014 10:56 AM

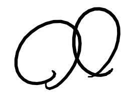
What is a stack?

0-stack (= sheaf)

stuff on open sets

= on int's of 2 opens

U V



α_u α_v $\alpha_u|_{u\cap v} = \alpha_v|_{u\cap v}$

1-stack (= stack)

stuff on open sets

identifications on int's of 2 opens

id's = on it's of 3 opens

U V

α_u α_v

W

α_w

$\varphi_{u,v} : \alpha_u|_{u\cap v} \to \alpha_v|_{u\cap v}$

$\varphi_{v,w}$ $\varphi_{u,w}$

$\varphi_{vw} \circ \varphi_{uv} = \varphi_{uw}$

2-stack

stuff on open sets α_u

id's on int's of 2 opens $\varphi_{u,v}$

id's of id's on 3 opens $\psi_{u,v,w} : \varphi_{vw} \circ \varphi_{uv} \xrightarrow{\sim} \varphi_{uw}$

id's of id's are = on 4 opens

∞-stacks

Basic example: assoc. to each open a top space.

<u>Ex</u>: X top space: $Shv(X)$ is a stack.

(means: to give a sheaf on X, "same as" giving sheaves $\mathcal{F}_{U_i}$

on cover $\{U_i \to X\}$, and id's $\varphi_{ij} : \mathcal{F}_{U_i}|_{U_i\cap U_j} \xrightarrow{\sim} \mathcal{F}_{U_j}|_{U_i\cap U_j}$

(means : to give a s... on an open cover $\{U_i \to X\}$, and id's $\varphi_{ij} : \mathcal{F}_{U_i}|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_{U_j}|_{U_i \cap U_j}$

s.t. $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$ on $U_i \cap U_j \cap U_k$)

<u>Ex</u> X a scheme in Zariski top, $Sch/(Zar/X)$

if $U \subset X$ Zar. open consider U schemes.

$U \longrightarrow U$-schemes

U-scheme given by $\{U_i \to U\}$

$f_i\ X_i \to U_i$ U_i-scheme each i,

$\varphi_{ij} : f_i^{-1}(U_i \cap U_j) \to f_j^{-1}(U_i \cap U_j)$

$+$ $\varphi_{jk}| \varphi_{ij}| = \varphi_{ik}|$

To make sense of this, want to describe a stack w/ is

$\{\text{Open}\} \rightsquigarrow$ "Sections" w/ notion of iso's.

(objects in site) Categories

also need restriction maps.

<u>Observation</u>/Question: Did we ever really have restriction maps anyways?

<u>Prototype</u>:

$$\begin{array}{c} X \\ \downarrow \text{family} \\ V \xrightarrow[f]{} U \end{array}$$

"restrict X to V"

"$X|_V$" $= X \times_U V = f^* X$

ugly issue: if we pick our favorite model for the fiber product then we actually won't generally have $g^* f^* X = (f \circ g)^* X$

$$\begin{tikzcd} & & X \arrow[d] \\ W \arrow[r, "g"'] & V \arrow[r, "f"'] & U \end{tikzcd}$$

but just $\uparrow$ canonical iso morphism.

Fibered Categories

Def $\mathcal{C}$ a category. A category over $\mathcal{C}$ is a pair (F, p) where F is a category & $p : F \to \mathcal{C}$ a functor.

ex: $\mathcal{C} = Zr/X$ $\quad F = \{ f : Y \to U \mid U \text{ open in } X \}$

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"] & Y \arrow[d, "f"] \\ V \arrow[r] & U \end{tikzcd}$$

morph = comm. diagram.

$$p : F \to \mathcal{C}$$
$$(f : Y \to U) \longmapsto U$$

Def If $F \xrightarrow{p} \mathcal{C}$ is a cat. over $\mathcal{C}$, then a morphism $\varphi : \xi \to \eta$ is called cartesian if for any $\zeta \in F$ & $\psi : \zeta \to \eta$, and h s.t.

$$\begin{tikzcd} p(\zeta) \arrow[r, "h"] \arrow[rr, bend right, "p(\psi)"'] & p(\xi) \arrow[r, "p(\varphi)"] & p(\eta) \end{tikzcd}$$

then $\exists!\ \lambda : \zeta \to \xi$ s.t. $\varphi \circ \lambda = \psi$ and $p(\lambda) = h$

$$\begin{tikzcd} \zeta \arrow[r, dashed, "\exists!"] \arrow[rr, bend left, "\psi"] & \xi \arrow[r, "\varphi"'] & \eta \end{tikzcd}$$

$$\begin{tikzcd}
\zeta \arrow[r, dashed, "\exists!"] \arrow[rr, bend left, "\psi"] \arrow[d, mapsto] & \xi \arrow[r, "\varphi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
p(\zeta) \arrow[r, "h"] \arrow[rr, bend right, "p(\psi)"'] & p(\xi) \arrow[r, "p(\varphi)"] & p(\eta)
\end{tikzcd}$$

In this case, we say that

ξ is <u>a</u> pullback of η along $p(\varphi)$ = "$p(\varphi)^* \eta$"

<u>Notation:</u> $p: F \to \mathcal{C}$ cat. over $\mathcal{C}$, we write $F(U)$ for cat w/ objects $\eta \in F$ s.t. $p(\eta) = U$ & w/ $\mathrm{Hom}_{F(U)}(\eta, \xi)$ = homs $\eta \xrightarrow{\varphi} \xi$ s.t. $p(\varphi) = id_U$

<u>Def:</u> $p: F \to \mathcal{C}$ is a fibred cat. if $\forall f: U \to V$ in $\mathcal{C}$ & $\eta \in F(V)$, $\exists$ cart. arrow $\varphi: \xi \to \eta$ s.t. $p(\varphi) = f$.

If $F, G \xrightarrow{p_F, p_G} \mathcal{C}$ fib. cats, then a morphism $g: F \to G$ is a functor s.t.

$$\begin{tikzcd}
F \arrow[rr, "g"] \arrow[dr, "p_F"'] & & G \arrow[dl, "p_G"] \\
& C &
\end{tikzcd}$$

1) $p_G \circ g = p_F$

2) g takes cart arrows to cart. arrows.

If $g, g': F \to G$ morph. of fibred cats, then a nat. trans. $\alpha: g \to g'$ is a base preserving nat. trans. if $\forall \xi \in F$, $\alpha(\xi): g(\xi) \to g'(\xi)$ then

$$p_G(\alpha(\xi)) = id_{p_F(\xi)}$$

... objects = morphisms of fibred cats (... arrows) morphs = nat. trans (base pt ...)

$\Rightarrow$ Fibred cats are a "2-category"

<u>Lem</u> If $p: F \to \mathcal{C}$ a fibred cat, and $\psi: \zeta \to \eta$ is any morphism in F, then can factor ψ as:

$$\begin{tikzcd} \zeta \arrow[r, "\lambda"] \arrow[rr, bend right, "\psi"'] & \xi \arrow[r, "\varphi"] & \eta \end{tikzcd}$$

where φ is cartesian &,
$\lambda \in F(p(\xi))$
$= F(p(\zeta))$

<u>Pf.</u> just look at a pullback of η along $p(\psi)$

$$\begin{tikzcd} \zeta \arrow[d, mapsto] \arrow[r, dashed, "\lambda"] \arrow[rr, bend left, "\psi"] & p(\psi)^*\eta \arrow[r, "\varphi"', "\text{cart.}"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ p(\zeta) \arrow[r, equal] & p(\zeta) \arrow[r, "p(\psi)"'] & p(\eta) \end{tikzcd}$$

<u>Lem</u> If $g: F \to G$ a morph. of fibred cats s.t. $\forall\, u \in \mathcal{C}$, $g_u: F(u) \to G(u)$ is fully faithful, then g is fully faithful

Pf: idea:

$$\begin{tikzcd} \mathrm{Hom}_F(\xi, \eta) \arrow[rr, "g"] \arrow[dr] & & \mathrm{Hom}_G(g(\xi), g(\eta)) \arrow[dl] \\ & \mathrm{Hom}_{\mathcal{C}}(p_F(\xi), p_F(\eta)) & \end{tikzcd}$$

$$\underset{h}{\in}$$

maps $\xi \to \eta$ above h $\Leftrightarrow$ maps $\xi \to h^*\eta \in F(p_F\xi)$

$g(\xi) \to g(\eta)$ " $-$ $\Leftrightarrow$ maps $g(\xi) \to h^* g(\eta) \in G(p_F\xi)$

$\square$