Lecture 7: fibered categories 2

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<u>Def</u> $g : F \to G$ a morphism of fibered cats over $\mathcal{C}$ is an <u>equivalence</u> if $\exists\, h : G \to F$, and base preserving nat isomorphisms $h \circ g \simeq id_F$ and $g \circ h \simeq id_G$

<u>Prop</u> $g : F \to G$ is an equiv ($/\mathcal{C}$) if and only if $\forall\, U \in \mathcal{C}$, $g(U) : F(U) \to G(U)$ is an equiv. of cats.

<u>Pf:</u> prev result $\Rightarrow$ g fully faithful

Construct h via: for each $y \in G$, $(p_G(y) = U)$

know that $F(U) \to G(U)$ is an equiv $\Rightarrow$ ess. surj

$\Rightarrow$ can choose $x \in F(U)$ s.t. $gx \simeq y$

define $h(y) = x$. standard argument (eq. of cats = ff + ess. surj)

<u>Ex</u> If $\mathcal{G}$ is a presheaf on $\mathcal{C}$, can define a fibered cat F as follows:

[<u>Def</u> A set is a category s.t. all morphisms are identities]

Objects of F are pairs (U, α), $\alpha \in \mathcal{G}(U)$

morphisms $Hom_F((U,\alpha),(V,\beta)) = \{ f \in Hom_{\mathcal{C}}(U,V) \mid \mathcal{G}(f)(\beta) = \alpha \}$

$\mathcal{F}(f)(\beta) = \alpha\}$

$$\begin{array}{ccc} F & & (U,\alpha) \\ \downarrow & & \downarrow \\ \mathcal{C} & & U \end{array} \qquad\qquad U \to V, \quad \alpha = \beta|_U, \ \beta$$

pullbacks? ✓

$$\begin{tikzcd} \overset{\mathcal{F}(f)(\alpha)}{\overset{\parallel}{(V, \alpha|_V)}} \arrow[r] \arrow[d] & (U,\alpha) \arrow[d] \\ V \arrow[r, "f"] & U \end{tikzcd}$$

Note:
the cats $F(U) = \mathcal{F}(U)$ (a set!)

Rem: this gives a functor from the cat. of presheaves $/\mathcal{C}$ to the (2)-cat of fibred categories $/\mathcal{C}$

Moreover:
Prop (3.2.8) We have an equiv. of cat's

$$\underline{\text{Presheaves}/\mathcal{C}} \longleftrightarrow \underline{\text{Categories fibred in Sets}}/\mathcal{C}$$

ex: this (2) subcat of fib. cats has no nontriv. 2-morph

2-Yoneda Lemma

Yoneda says: If $\mathcal{C}$ a cat, $\hat{\mathcal{C}}$ presheaves on $\mathcal{C}$

we have functor $\mathcal{C} \to \widehat{\mathcal{C}}$

$$X \longmapsto h_X = (Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X))$$

$$\text{Magic} = \left(\mathrm{Hom}_{\widehat{\mathcal{C}}}(h_X, \mathcal{F}) \cong \mathcal{F}(X) \right)$$

<u>2-Yoneda:</u> $\mathcal{C} \longrightarrow \widehat{\mathcal{C}} \simeq \underline{\text{Cats fibred in Sets}}/\mathcal{C}$

$$\hookrightarrow \text{Fib. Cats}/\mathcal{C}$$

$$\mathrm{HOM}_{\mathcal{C}}((\mathcal{C}/X), F) \xrightarrow{\sim} F(X)$$

<u>Lemma</u>: $\mathcal{C}/X$ = fibered cat. associated to the presheaf h_X.

$$\mathcal{C}/X = F_{h_X}$$

<u>Pf</u>: $\mathrm{Ob}(F_{h_X}) = (U, \alpha) = (U, \alpha : U \to X) = \mathrm{Ob}(\mathcal{C}/X)$

$\alpha \in h_X(U) = U \to X$

$$\mathcal{C}/X \to \mathcal{C}$$

$$(U \mapsto X) \longmapsto U \qquad \text{exercise!} \qquad \square$$

<u>Pf of Y.II</u>. $F(X) \longrightarrow \mathrm{HOM}_{\mathcal{C}}((\mathcal{C}/X), F)$

$x \mapsto ((f : U \to X) \overset{\rho_x}{\longmapsto} f^*x \in F(U))$

choose one!

$$\mathcal{F}(X) \ni x \rightsquigarrow (U \to X) \rightsquigarrow \mathcal{F}(U)$$

$x \longmapsto x|_U$

on morphisms, $\rho_x(\alpha : U \to V)$

$$\begin{tikzcd} b^*x \arrow[rrr, dashed, bend left, "\exists! = \rho_x(\alpha)"] \arrow[d, "\beta"'] & U \arrow[r] \arrow[d, "b"] & V \arrow[d, "c"] & c^*x \arrow[d, "\gamma"] & \mathcal{C} \\ x & X & X & x & \end{tikzcd}$$

Why YII is good.

Want to describe stacks as "generalized sheaves" $\mathcal{X}$

assoc. to scheme $U \rightsquigarrow \mathcal{X}(U)$ = families over U
$\qquad\qquad\qquad\qquad\qquad \text{Hom}(U, \mathcal{X})$ category

$V \to U$ can pullback (non canonically) $\mathcal{X}(U) \dashrightarrow \mathcal{X}(V)$

on the other hand, schemes also should corresp. to stacks

$$\text{Hom}_{\text{stacks}}(\underbrace{\text{Stack } U}_{F_{h_U} = (\mathcal{C}/U)}, \mathcal{X}) = \mathcal{X}(U)$$

$\uparrow$ fibral cat. over U of fibred cat $\mathcal{X}$

Want: $\text{HOM}_{\text{"Sch"}}((\mathcal{C}/U), \mathcal{X}) = \mathcal{X}(U)$ is 2-Yoneda!

Skip 3.3 - splittings of fibred cats

Groupoids

Def A groupoid is a category s.t. all morphisms are isomorphisms.

Def A fibred cat $F \xrightarrow{p} \mathcal{C}$ is a cat. fibred in groupoids if $\forall U \in \mathcal{C}$, $F(U)$ is a groupoid.

"Recall" If $G, F \to \mathcal{C}$ are fibred in sets then $HOM_{\mathcal{C}}(F,G)$ is a set.

Prop If $G, F \to \mathcal{C}$ are fibred in groupoids, then $HOM_{\mathcal{C}}(F,G)$ is a groupoid.

Pf idea: if $f, g \in HOM_{\mathcal{C}}(F,G)$ and $\alpha : f \to g$ is a morphism (base preserving natural trans), then α is the data of $\forall x \in F$

$$\alpha(x) : f(x) \to g(x) \in G(p_F(x))$$

set $\beta : g \to f$ s.t. $\beta(x) = \alpha(x)^{-1} : g(x) \to f(x)$

this is an inverse to α. $\square$