

Def $g: F \rightarrow G$ a morphism of fibered cats over \mathcal{C} is an equivalence if $\exists h: G \rightarrow F$, and base preserving nat isomorphisms $h \circ g \cong \text{id}_F$ and $g \circ h \cong \text{id}_G$

Prop $g: F \rightarrow G$ is an equiv (1/c) if and only if $\forall u \in \mathcal{C}$, $g(u): F(u) \rightarrow G(u)$ is an equiv. of cats.

Pf: prev result $\Rightarrow g$ fully faithful

Construct h via: for each $y \in G$, ($p_G(y) = u$)
 know that $F(u) \rightarrow G(u)$ is an equiv \Rightarrow ess. surj
 \Rightarrow can choose $x \in F(u)$ s.t. $gx \cong y$

define $h(y) = x$. standard argument (eq. of cats = ft + ess. surj)

Ex If \mathcal{G} is a presheaf on \mathcal{C} , can define a fibered cat F as follows:

Def A set is a category s.t. all morphisms are identities

Objects of F are pairs (u, α) , $\alpha \in \mathcal{G}(u)$

morphisms $\text{Hom}_F((u, \alpha), (v, \beta)) = \{ f \in \text{Hom}_{\mathcal{C}}(u, v) \mid \alpha(u) \circ f = \beta \}$

map \dots \dots \dots

$$\exists (A/\rho) = \alpha$$

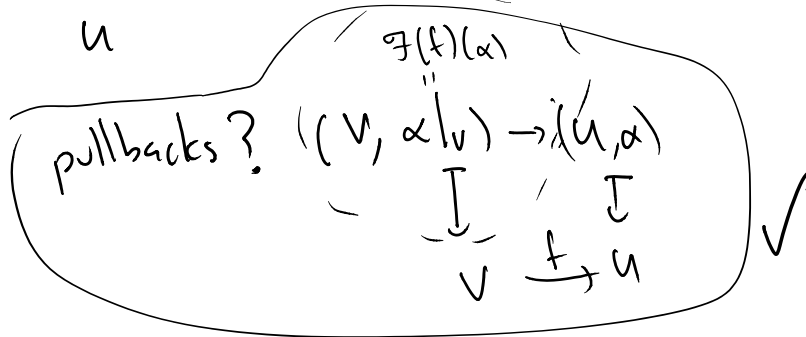
$F(U, \alpha)$

$U \rightarrow V$

$\downarrow \quad \downarrow$

$\alpha = \beta|_U$

$C \quad U$



Note:

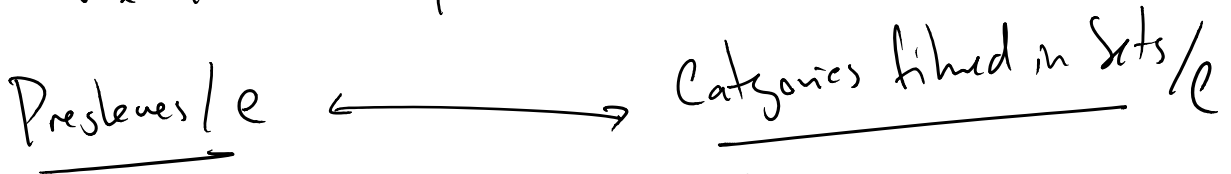
the cats $F(U) = \exists(U)$ (a set!)

Rem: this gives a functor from the cat. of presheaves $/C$ to the (2)-cat of fibred categories $/C$

More:

Prop (3.2.8)

We have an equiv. of cats



ex: this subset of fib. cats (2) has no nat. trans. 2-morph.

2- Yoneda Lemmas

Yoneda says: If C a cat, \hat{C} presheaves on C

we have functor $\mathcal{C} \rightarrow \hat{\mathcal{C}}$
 $X \mapsto h_X = (Y \mapsto \text{Hom}_{\mathcal{C}}(Y, X))$

$$\text{Magic} = \left(\text{Hom}_{\hat{\mathcal{C}}}(h_X, \mathcal{F}) \cong \mathcal{F}(X) \right)$$

2-Yoneda: $\mathcal{C} \rightarrow \hat{\mathcal{C}} \simeq \underline{\text{Cats fibred in Sets}/e}$
 $\hookrightarrow \text{Fib. Cats}/e$

$$\text{Hom}_{\mathcal{C}}(\mathcal{C}/X, F) \xrightarrow{\sim} F(X)$$

lemma: $\mathcal{C}/X =$ fibred cat. associated to the presheaf h_X .

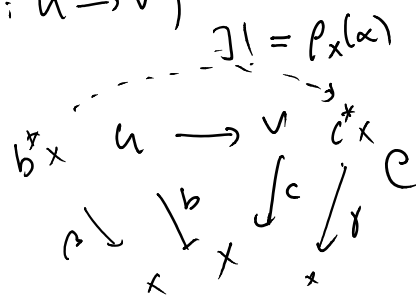
Pf: $\text{Ob}(F_{h_X}) = (U, \alpha) = (U, \alpha: U \rightarrow X)$ F_{h_X}
 $\alpha \in h_X(U)$ " $\text{Ob}(\mathcal{C}/X)$
 $U \rightarrow X$

$\mathcal{C}/X \rightarrow \mathcal{C}$
 $(U \rightarrow X) \rightarrow U$ D
 exercise!

Pf of Y.II. $F(X) \rightarrow \text{Hom}_{\mathcal{C}}(\mathcal{C}/X, F)$
 \downarrow \downarrow
 X $(f: U \rightarrow X) \xrightarrow{p_X} f^*_X \in F(U)$
 \uparrow
 choose one!

$$\mathcal{F}(X) \ni x \rightsquigarrow (U \rightarrow X) \rightsquigarrow \mathcal{F}(U)$$

on morphisms, $p_x(\alpha: u \rightarrow v)$ $x \mapsto x|_u$



Why YII is good.

want to describe stacks as "generalized schemes" \mathcal{X}

assoc. to scheme $u \mapsto \mathcal{X}(u) = \text{families over } u$
 $\text{Hom}(u, \mathcal{X})$ category

$v \rightarrow u$ can pullback (non canonically) $\mathcal{X}(u) \dashrightarrow \mathcal{X}(v)$

on the other hand, schemes also should corresp. to stacks

$$\text{Hom}_{\text{Stacks}}(\underbrace{\text{Stack } u, \mathcal{X}}_{\substack{F_{hu} \\ \text{"} \\ (C/u)}}) = \mathcal{X}(u)$$

↑
fibrat cats over u of fibrat cat \mathcal{X}

want: $\text{HOM}_{\text{Sch}}((C/u), \mathcal{X}) = \mathcal{X}(u)$ is 2-Yoneda!

Skip 3.3 - splittings of fibrat cats

Groupoids

Def A groupoid is a category s.t. all morphisms are isomorphisms.

Def A fibred cat $F \xrightarrow{P} \mathcal{C}$ is a cat. fibred in groupoids if $\forall u \in \mathcal{C}$, $F(u)$ is a groupoid.

"Recall" If $G, F \rightarrow \mathcal{C}$ are fibred in sets then $\text{HOM}_{\mathcal{C}}(F, G)$ is a set.

Prop If $G, F \rightarrow \mathcal{C}$ are fibred in groupoids, then $\text{HOM}_{\mathcal{C}}(F, G)$ is a groupoid.

PF idea: if $f, g \in \text{HOM}_{\mathcal{C}}(F, G)$ and $\alpha: f \rightarrow g$ is a morphism, then α is the data of $\forall x \in F$ (base preserving natural trans)

$$\alpha(x): f(x) \rightarrow g(x) \in G(P_F(x))$$

set $\beta: g \rightarrow f$ s.t. $\beta(x) = \alpha(x)^{-1}: g(x) \rightarrow f(x)$
this is an inverse to α . \square