Lecture 8: fibered categories 3

Thursday, September 11, 2014   11:02 AM

Category: Sets $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$

objects arrows

$s, t : \mathcal{C}_1 \longrightarrow \mathcal{C}_0$ (source & target)

$\varepsilon : \mathcal{C}_0 \longrightarrow \mathcal{C}_1$ (identity)

$m : \underbrace{\mathcal{C}_1 \times_{s, \mathcal{C}_0, t} \mathcal{C}_1}_{} \longrightarrow \mathcal{C}_1$

fiber product

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d, "t"] \\ \mathcal{C}_1 \arrow[r, "s"'] & \mathcal{C}_0 \end{tikzcd}$$

s.t.

ε is id

start & end: $s \circ \varepsilon = t \circ \varepsilon = id_{\mathcal{C}_0}$, $s \circ m = s \circ pr_2$, $t \circ m = t \circ pr_1$ maps from fiber prod

id:

$$\begin{tikzcd} & \mathcal{C}_1 \times_{s, \mathcal{C}_0} \mathcal{C}_0 \arrow[dr, "id \times \varepsilon"] & & \\ \mathcal{C}_1 \arrow[ur, equal] \arrow[dr, equal] \arrow[rr, bend left, "id_{\mathcal{C}_1}"] & & \mathcal{C}_1 \times_{s, \mathcal{C}_0, t} \mathcal{C}_1 \arrow[r, "m"] & \mathcal{C}_1 \\ & \mathcal{C}_0 \times_{t, \mathcal{C}_0} \mathcal{C}_1 \arrow[ur, "\varepsilon \times id"'] & & \end{tikzcd}$$

commutes

assoc:

$$C_1 \times_{s,C_0,t} C_1 \times_{s,C_0,t} C_1 \overset{m\times id}{\underset{id\times m}{\rightrightarrows}} C_1 \times_{s,C_0,t} C_1 \xrightarrow{m} C_1$$

commutes.

Given a cat $\mathcal{C}$, a cat. object in $\mathcal{C}$ is a pair of objects C_0, C_1 w/ morphisms s, t, ε, m as above.

Similarly, groupoid objects in a category $\mathcal{C}$ as above, but w/ extra morphism $i: C_1 \to C_1$ "inverse"

Groupoid = groupoid object in Set

eq. rel: $R \overset{i}{\subset} S \times S$

$$\underset{C_1}{\overset{\shortparallel}{R}} \overset{pr_1 \circ i}{\underset{pr_2 \circ i}{\rightrightarrows}} \underset{C_0}{\overset{\shortparallel}{S}}$$

$(s,t) \in R$ $s \sim t$ well def mult

$s \sim t$ $t \sim u \Rightarrow s \sim u$

$s \sim s$ $s \sim t \Rightarrow t \sim s$

id inverses

note: if $\exists$ arrow $s \to t$ then it is unique.

$\rightsquigarrow$ ~~cat~~ a groupoid.

altrnately eq. rel = gpoid w/ all "arrows unique"

$\forall a \in \mathcal{C}_0, \ \mathrm{Aut}_{\mathcal{C}}(a) = id_a$

altrnately : Cat is an eq rel if it is eq. to a set.

$\boxed{\text{excercise}}$

Given a category $\mathcal{C}$ & a groupoid object $X = (X_0, X_1, \ldots)$ in $\mathcal{C}$, we get a cat fibered in groupoids $/\mathcal{C}$!

(Recall if $Y \in \mathcal{C}$, $\left(\mathcal{C}/Y\right)$ fibered cat)

$$\begin{array}{c} (\mathcal{C}/Y) \\ \downarrow \\ \mathcal{C} \end{array}$$

$$\begin{array}{c} \{X_0/X_1\} \\ \downarrow \\ \mathcal{C} \end{array}$$

for $U \in \mathcal{C}$, $\{X_0/X_1\}(U) = (X_0(U), X_1(U), \ldots)$

groupoid

$X_0(U) = \mathrm{Hom}(U, X_0) \ldots$

i, given $V \xrightarrow{f} U$

get $\mathrm{Hom}(U, X_0) \xrightarrow{f^*} \mathrm{Hom}(V, X_0)$

$\{X_0/X_1\}$ objects are (U, u) $U \in \mathcal{C}$, $u \in X_0(U)$

morphisms

$\mathrm{Hom}((U,u),(V,v)) =$

map $f: u \to v$ & an isom

$(X, (u))$

$u \to f^* v$

$$\begin{array}{l} F \\ \downarrow p \\ \mathcal{C} \end{array}$$ fibred cat, $x, x' \in F(X)$

$$\underline{\mathrm{Isom}}(x,x') : (\mathcal{C}/X)^{op} \longrightarrow \text{"Set"}$$

$$\mathrm{Isom}(x,x')(X) = \mathrm{Hom}_{F(X)}(x,x')$$

$$\underline{\mathrm{Isom}}(x,x') \ni (f, U), \quad f : y \xrightarrow{\sim} y' \in F(U)$$

$\underset{\varphi}{U \to X}$ where y is a pullback of x over φ

$$\begin{array}{c} \underline{\mathrm{Isom}}(x,x') \\ \downarrow \\ (\mathcal{C}/X) \end{array}$$

$$\begin{tikzcd} y \arrow[r] \arrow[d, dash] & x \arrow[d, dash] \\ U \arrow[r, "\varphi"] & X \end{tikzcd}$$

y' pullback of x'

morphism

$$\begin{tikzcd} y \arrow[r, "\sim", "f"'] \arrow[d, "F(U)"'] & y' \arrow[d, "F(U)"] \\ z \arrow[r, "\sim", "g"'] & z' \end{tikzcd}$$

$$\begin{array}{c} f, U \\ \downarrow \\ g, U \end{array}$$

but Isom is a functor!