Lecture 9: fibered categories 4/ descent 1

Tuesday, September 16, 2014 11:00 AM

Fiber products of fibred cat's.

Fiber products of schemes $X, Y \to S$

$$\begin{tikzcd} Z \arrow[dr, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

$$\begin{array}{c} \mathrm{Hom}(Z, X\times_S Y) \quad h_{X\times_S Y}(Z) \\ \downarrow \\ \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y) \\ \downarrow \downarrow \\ \mathrm{Hom}(Z, S) \end{array}$$

really says: $h_{X\times_S Y}(Z) = h_X(Z) \times_{h_S(Z)} h_Y(Z)$

To define fiber prods of fibred cat's (fibred in gpoids)

need fiber prods of groupoids

2 fiber prod of categories:

$\mathcal{C}_1, \mathcal{C}_2$ functors $F_i : \mathcal{C}_i \to \mathcal{C}$

$\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2$ objects: (c_1, c_2, φ) $c_i \in Ob(\mathcal{C}_i)$

$\varphi : F_1 c_1 \xrightarrow{\sim} F_2 c_2$

$\in \mathrm{Hom}_{\mathcal{C}}(F_1 c_1, F_2 c_2)$

$\mathrm{Hom}_{\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2}((c_1, c_2, \varphi), (d_1, d_2, \psi))$

are maps $c_i \xrightarrow{g_i} d_i$ s.t.

$$\begin{tikzcd} F_1 c_1 \arrow[r, "F_1 g_1"] \arrow[d, "\varphi"] & F_1 d_1 \arrow[d, "\psi"] \\ F_2 c_2 \arrow[r, "F g_2"'] & F_2 d_2 \end{tikzcd}$$ commutes.

Univ. property:

$$\begin{tikzcd} C_1 \times_C C_2 \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & C_1 \arrow[dr, "F_1"] & \\ & C_2 \arrow[r, "F_2"'] & C \arrow[from=1-2, to=2-2, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

$$\alpha(c_1, c_2, \varphi) : F_1 c_1 \xrightarrow{\sim} F_2 c_2$$
$$\overset{\|}{\varphi}$$

Further if

$$\begin{tikzcd} \mathcal{H} \arrow[r, "p_1"] \arrow[dr, "p_2"'] & C_1 \arrow[dr, "F_1"] & \\ & C_2 \arrow[r, "F_2"'] & C \arrow[from=1-2, to=2-2, phantom, "\Downarrow\beta"] \end{tikzcd}$$

then:

define $\lambda : \mathcal{H} \to C_1 \times_C C_2$ via

$$\lambda(h) = (p_1 h, p_2 h, \beta(h))$$

so that

$$\begin{tikzcd} \mathcal{H} \arrow[r, "p_i"] \arrow[dr, "\lambda"'] & C_i \\ & C_1 \times_C C_2 \arrow[u] \end{tikzcd}$$

$p_i h$

commutes

h

$(p_1 h, p_2 h, \ )$

and

$$\begin{matrix} F_1 p_1 h & = F_1 \pi_1 \lambda(h) \\ \downarrow \beta(h) & \downarrow \alpha(\lambda(h)) \\ F_2 p_2 h & = F_2 \pi_2 \lambda(h) \end{matrix}$$

want $\beta(h) = \alpha(\lambda(h))$

show diagrams commute

i.e. $\exists!\ \lambda$ s.t. above dgrams commute

Suppose $\mathcal{C}$ site,

$$\begin{tikzcd} & F_1 \arrow[d, "c"] \\ F_2 \arrow[r, "d"] & F_3 \end{tikzcd}$$

diagram of morphs of cats fibred in groupoids over $\mathcal{C}$.

we would like to understand fib. cats G w/ morphs$/\mathcal{C}$

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & F_1 \\ & F_2 \end{tikzcd}$$

and isom $\gamma : c\circ\alpha \to d\circ\beta$
$\in$ $HOM_{\mathcal{C}}(G,F_3)$

this together gives an elmt of fiber prod cat:

$$(\alpha,\beta,\gamma) \in HOM_{\mathcal{C}}(G,F_1) \times_{HOM_{\mathcal{C}}(G,F_3)} HOM_{\mathcal{C}}(G,F_2)$$

" $\|$ "

$$HOM_{\mathcal{C}}(G, F_1 \times_{F_3} F_2)$$

Given our G, for every $H \to G$, get a new triple

$$HOM_{\mathcal{C}}(H,G) \longrightarrow HOM_{\mathcal{C}}(H,F_1) \times_{HOM_{\mathcal{C}}(H,F_3)} HOM_{\mathcal{C}}(H,F_2)$$

mor $H \to G \rightsquigarrow$

Claim $\exists$ "$G = F_1 \times_{F_3} F_2$" s.t. is an equiv all H.

- If G, G' both universal as above then $\exists$ an equiv $F : G \longrightarrow G'$ of fibred cats

Construction:

G objects (x_1, x_2, σ) $\quad x_i \in F_i \quad p_1 x_1 = p_2 x_2$

$(F_1 \times_{F_3} F_2)$ $\qquad \sigma : c x_1 \xrightarrow{\sim} d x_2$ in $F_3(p_1(x_1))$

morphisms $(x_1', x_2', \sigma') \longrightarrow (x_1, x_2, \sigma)$

given by

$$\begin{tikzcd} x_i' \arrow[r, "f_i"] \arrow[d, mapsto] & x_i \arrow[d, mapsto] \\ p_i(x_i') \arrow[r, "p_i(f_i)"'] & p_i(x_i) \end{tikzcd} \qquad p_1 f_1 = p_2 f_2$$

and such that

$$\begin{tikzcd} c(x_1') \arrow[r, "c(f_1)"] \arrow[d, "\sigma'"] & c(x_1) \arrow[d, "\sigma"] \\ d(x_2') \arrow[r, "d"'] & d(x_2) \end{tikzcd}$$ commutes $\triangleright$

Descent & the stack Condition

Main first fact : If $\begin{matrix} X \\ \downarrow \pi \\ S \end{matrix}$ a scheme $/S$ then h_X is sheaf in the fppf top$/S$.

$$\mathrm{Hom}(U, X) \longrightarrow \prod \mathrm{Hom}(U_i, X) \rightrightarrows \prod \mathrm{Hom}(U_i \times_U U_j, S)$$

$\{U_i \to U\}$ fppf cov exact.