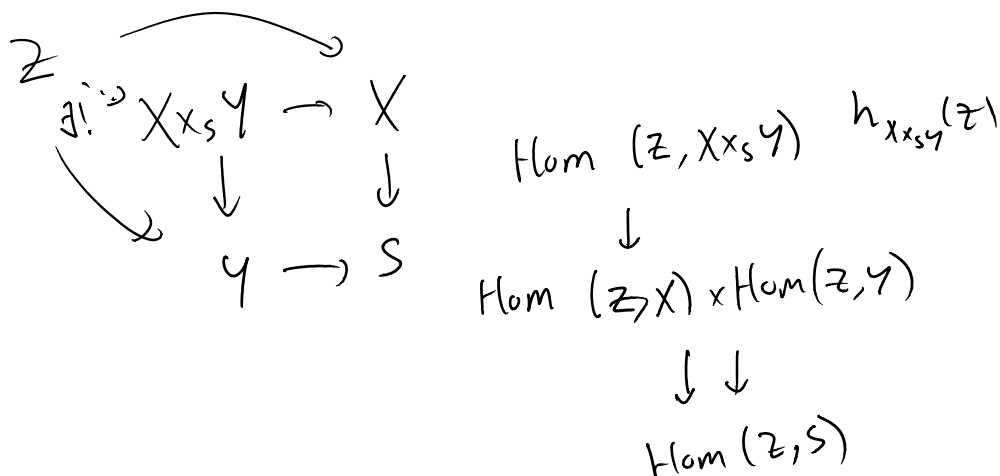


Fiber products of fibered cat's.

Fiber products of schemes  $X, Y \rightarrow S$



really says:  $h_{X \times_S Y}(z) = h_X(z) \times_{h_S(z)} h_Y(z)$

to define fiber products of fibered cat's (fibered in groupoids)  
 need fiber products of groupoids

2 fiber prod of categories:

$C_1, C_2$  functors  $F_i: C_i \rightarrow C$

$C_1 \times C_2$  objects:  $(c_1, c_2, \varphi)$   $c_i \in \text{Ob}(C_i)$   
 $\varphi: F_1 c_1 \xrightarrow{\sim} F_2 c_2$

$\text{Hom}_{C_1 \times C_2}((c_1, c_2, \varphi), (d_1, d_2, \psi)) \xleftarrow{\sim} \text{Hom}_C(F_1 c_1, F_2 c_2)$

are maps  $c_i \xrightarrow{g_i} d_i$  s.t.

$$\begin{array}{ccc}
 F_1 c_1 & \xrightarrow{F_1 g_1} & F_1 d_1 \\
 \downarrow \varphi & & \downarrow \psi \text{ commutes.} \\
 F_2 c_2 & \xrightarrow{F_2 g_2} & F_2 d_2
 \end{array}$$

Unique property:

$$\begin{array}{ccccc}
 C_1 \times C_2 & \xrightarrow{\pi_1} & C_1 & \xrightarrow{F_1} & C \\
 & \searrow & \downarrow \alpha & \searrow & \\
 & & C_2 & \xrightarrow{F_2} & C \\
 & \swarrow & & & \\
 & & C_2 & & \\
 & \xrightarrow{\pi_2} & & & 
 \end{array}$$

$\alpha(c_1, c_2, \varphi): F_1 c_1 \xrightarrow{\sim} F_2 c_2$   
 $\downarrow \varphi$

Further if  $H \xrightarrow{p_1} C_1 \xrightarrow{F_1} C$  then:

$$\begin{array}{ccccc}
 H & \xrightarrow{p_1} & C_1 & \xrightarrow{F_1} & C \\
 & \searrow & \downarrow \beta & \searrow & \\
 & & C_2 & \xrightarrow{F_2} & C \\
 & \swarrow & & & \\
 & & C_2 & & \\
 & \xrightarrow{p_2} & & & 
 \end{array}$$

define  $\lambda: H \rightarrow C_1 \times C_2$  via

$$\lambda(h) = (p_1 h, p_2 h, \beta(h))$$

so that

$$\begin{array}{ccc}
 H & \xrightarrow{p_i} & C_i \\
 \downarrow \lambda & & \uparrow \\
 H & \xrightarrow{\lambda} & C_1 \times C_2
 \end{array}$$

$p_i h$   
 commutes  
 $(p_1 h, p_2 h, \beta(h))$

and

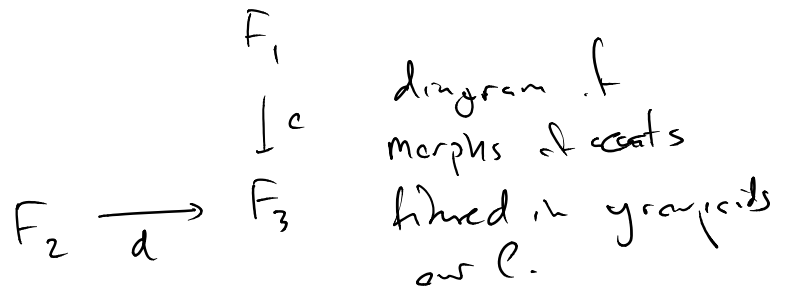
$$\begin{array}{ccc}
 F_1 p_1 h & = & F_1 \pi_1 \lambda(h) \\
 \downarrow \alpha(h) & & \downarrow \alpha(\lambda(h)) \\
 F_2 p_2 h & = & F_2 \pi_2 \lambda(h)
 \end{array}$$

want  $\beta(h) = \alpha(\lambda(h))$

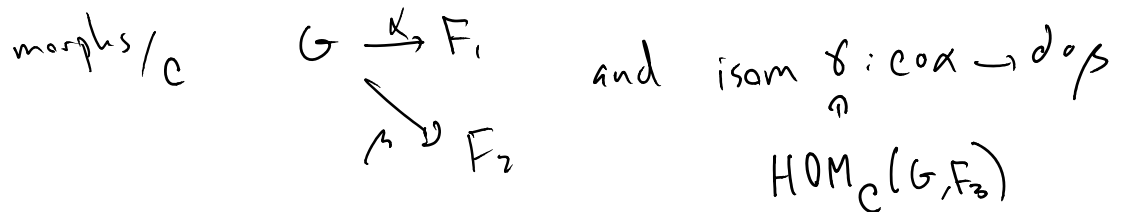
$\therefore$  show diagrams commute

i.e.  $\exists!$  s.t. above diagrams commute

Suppose  $\mathcal{C}$  site,



we would like to understand fib. csts  $G$  w/



this together gives an elab of fib. prod csti:

$$(\alpha, \beta, \delta) \in \text{HOM}_{\mathcal{C}}(G, F_1) \times_{\text{HOM}_{\mathcal{C}}(G, F_3)} \text{HOM}_{\mathcal{C}}(G, F_2)$$

"||"

$$\text{HOM}_{\mathcal{C}}(G, F_1 \times_{F_3} F_2)$$

Given our  $G$ , for every  $H \rightarrow G$ , get a new triple

$$\text{HOM}_{\mathcal{C}}(H, G) \rightarrow \text{HOM}_{\mathcal{C}}(H, F_1) \times_{\text{HOM}_{\mathcal{C}}(H, F_3)} \text{HOM}_{\mathcal{C}}(H, F_2)$$

mor  $H \rightarrow G \rightsquigarrow$

• Claim  $\exists$  " $G = F_1 \times_{F_3} F_2$ " s.t. is an equiv. all  $H$ .

- If  $G, G'$  both universal as above then  $\exists$  an equiv  $F: G \rightarrow G'$  of fibered cats

Construction:

$$G \text{ objects } (x_1, x_2, \sigma) \quad x_i \in F_i \quad p_1 x_1 = p_2 x_2$$

$$(F_1 x_1 F_2)$$

$$\sigma: c x_1 \xrightarrow{\sim} d x_2 \text{ in } F_3(p_1(x_1))$$

morphisms  $(x'_1, x'_2, \sigma') \rightarrow (x_1, x_2, \sigma)$

given by

$$x'_i \xrightarrow{f_i} x_i$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ p_1(x'_i) & \rightarrow & p_1(x_i) \end{array}$$

$$\uparrow p_i(f_i)$$

$$p_1 f_1 = p_2 f_2$$

and such that

$$c(x'_1) \xrightarrow{c(f_1)} c(x_1)$$

$$\downarrow \sigma'$$

$$\downarrow \sigma$$

commutes

$$d(x'_2) \xrightarrow{d} d(x_2)$$

$\square$

Descent of the stack condition

Main first fact: If  $X$  a scheme /  $S$  then  $h_X$  is stack in the fppt top /  $S$ .

$$\Pi_{i \in I} (h_X) \rightarrow \prod \text{Hom}(U_i, X) \xrightarrow{\sim} \prod \text{Hom}(U_i, X \times_{U_j} U_j, S)$$

$\{U_i \rightarrow U\}$  is an exact.